ManyVal '10 - Beyond algebraic semantics: bridging intended and formal interpretations of many-valued logics

May 3–5, 2010

Università degli Studi dell'Insubria
Varese, Italy

Contents

Invited talks 1

Degrees of Truth and Epistemic States by *Didier Dubois* 1
A Generalization of Giles's Game by *Chris Fermüller* 2
Belief Functions, Möbius Transform and Integral Representations on MV-algebras by *Tomáš Kroupa* 3
Admissible Rules of Many-Valued Logics by *George Metcalfe* 4
Toward a logic for imprecise probabilities by *Franco Montagna* 5
Notes on the foundations of fuzzy logic by *Thomas Vetterlein* 6

Contributed talks 8

Rough Sets and Many Valued Logics by *Davide Ciucci* 8
Extended-order algebras and fuzzy implicators by *Maria Emilia Della Stella* 11
A characterization of admissible Łukasiewicz assessments through t-SMV-algebras by *Martina Fedel* 12
MV-semirings and their sheaf representation by *Anna Rita Ferraioli* 14
Towards a betting interpretation for necessity measures of many-valued events by *Tommaso Flaminio* 16
Characterization of functions among lattice-valued relations by *Cosimo Guido* 19
Modelling degrees of truth: A case for objective probability by *Hykel Hosni* 20
Central limit theorem on MV-algebras by *Jana Kelemenová* 22
Individual ergodic theorem on Kôpka D-posets by *Lenka Lašová* 25
On counting MV algebras by *Giacomo Lenzi* 28
On unit intervals in L-spaces by *Ioana Leustean* 30
From Linguistic Models of Vagueness to t-Norm Based Fuzzy Logic by *Christoph Roschger* 32
Semiring and semimodule issues in MV-algebras by *Ciro Russo* 34
Omitting type theorems for Łukasiewicz logic by *Luca Spada* 37
Unitary Unification in Subvarieties of WNM-Algebras by *Diego Valota* 40

Degrees of Truth and Epistemic States

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In quite a few, sometimes influential, works dealing with knowledge representation, there is a temptation to extend the truth-set underlying a given logic with values expressing ignorance and contradiction. It then leads to some truth-functional many-valued logic different from the original one, but often using the same syntax. This is the case for instance with partial logic and Belnap bilattice logic with respect to classical logic. It is found again in interval-valued, and type two extensions of fuzzy sets.

This talk insists that neither ignorance nor contradiction cannot be viewed as additional truth-values nor processed in a truth-functional manner, and that doing it leads to weak or debatable uncertainty handling approaches. A similar difficulty is also found in three- valued logics of rough sets.

In fact, partial ignorance and contradiction are meta-level notions, like deduction and consistency and refer to epistemic states. When using the same language for modeling epistemic and objective notions, there is no way to tell the case of not believing a proposition from believing its contrary, not to tell the case of knowing that one among several propositions is true from the case of knowing that their disjunction is true. Modal logics have been used to handle epistemic notions in a more satisfactory way, but the Kripke semantics based on relations over possible worlds are perhaps unnecessarily complex to handle plain notions of epistemic states. A simpler semantics can be based on sets of epistemic states understood as non-empty subsets of interpretations.

We suggest that, in order to handle epistemic notions at the language level, we need two-tiered systems where one logic (typically propositional logic) describing events is encapsulated by another one (that can be many-valued) describing beliefs or testimonies. Such an approach paves the way to a general approach to logics of uncertainty, along the lines suggested by Esteva, Godo and Hájek casting probability, possibility or belief functions within a suitable multiple-valued setting, where beliefs are graded, but events remain Boolean.

References

[1] M. Banerjee, D. Dubois. A simple modal logic for reasoning about revealed beliefs - European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2009), Verona (Italy), Claudio Sossai, Gaetano Chemelle (Eds.), Springer, LNAI 5590, p. 805-816, 2009.

[2] D. Dubois: On Ignorance and Contradiction Considered as Truth-Values. Logic Journal of the IGPL 16(2): 195-216 (2008)

[3] D. Dubois, Degrees of Truth, Ill-Known Sets and Contradiction. In : Foundations of Reasoning under Uncertainty, (Bouchon-Meunier, B.; Magdalena, L.; Ojeda-Aciego, M.; Verdegay, J.-L.; Yager, R.R., Eds.) Studies in Fuzziness and Soft Computing, Volume 249, Springer, 2010, pp. 65-83.

[4] D. Dubois, F. Esteva, L. Godo, H. Prade. Fuzzy-set based logics - An history-oriented presentation of their main developments. In : Handbook of The history of logic. Dov M. Gabbay, John Woods (Eds.), V. 8, The many valued and nonmonotonic turn in logic, Elsevier, 2007, p. 325-449.

[5] D. Dubois, H. Prade. Possibility theory, probability theory and multiple-valued logics: A clarification. Annals of Mathematics and Artificial Intelligence. 32, 35-66, 2001.

A Generalization of Giles's Game

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Already in the 1970s Robin Giles presented a combination of a dialogue game and a particular betting scheme that characterizes Lukasiewicz logic. Variants of this type of game semantics for other important fuzzy logics, including Goedel and Product logic have been described in the literature. However these versions of the game suffer from major drawbacks. In particular the one-one relation between risk values and truth values of Giles's game is lost. We will discuss a way to generalize Giles's game in a more conservative, yet at the same time also more open and systematic manner. Quite general conditions on the evaluation of final states and on the form of dialogues rules are stated. These conditions turn out to be sufficient to guarantee that optimal strategies for the proponent of a formula correspond to a truth functional evaluation in a many valued logic over some subset of the reals as truth values. We will also discuss to which extend our conditions are necessary to characterize a many valued logic.

Belief Functions, Möbius Transform and Integral Representations on MV-algebras

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Probability on MV-algebras was developed in the papers by Mundici and Riečan. Among many investigated facets of this theory count conditiong or bookmaking over events represented by formulas in infinite-valued Łukasiewicz propositional calculus. A characterization relating many-valued probability to classical probability on Boolean algebras is the following: probability (state) on an MV-algebra is just Lebesgue integral with respect to a uniquely determined Borel probability measure on the maximal ideal space.

The condition of additivity from the definition of state can be relaxed to study more general real functionals on MV-algebras. Non-additive set functions, such as belief functions, possibility measures and upper probabilities, appear already on Boolean algebras in connection with game theory or statistics. The whole class of such imprecise probabilities on MV-algebras is studied in the recent paper by Fedel, Keimel, Montagna and Roth.

In this contribution we will single out *belief functions* on MV-algebras. In particular, we will show that every belief function

- is *totally monotone* with respect to the lattice reduct of the MV-algebra,
- is characterized through its *Möbius transform* which is a state on a certain MV-algebra constructed on the ideal space,
- is represented by *Choquet integral* with respect to a unique belief function on the maximal ideal space of the MV-algebra.

Admissible Rules of Many-Valued Logics

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Investigations of logical systems – in particular, many-valued logics – typically tend to focus on derivability. However, it can also be interesting and useful to "move up a level" and consider admissible rules of the system: the rules under which the set of theorems is closed. Whereas derivable rules (members of the consequence relation) may be thought of as providing an internal description of a logic, admissible rules provide an external perspective, describing properties. In algebra, such rules correspond to quasi-equations holding in free algebras, while from a computer science perspective, admissibility is intimately related to, and in certain cases may be reduced to, equational unification. For classical logic, derivability and admissibility coincide: the logic is structurally complete. However, for many non-classical logics – in particular, important intermediate, modal, many-valued, and substructural logics – this is no longer the case, and interesting questions arise as to the decidability, complexity, (finite) axiomatizability, and characterizations of their admissible rules (see, e.g., [3, 2, 1]).

The aim of this talk will be to explore the landscape of admissible rules and structural completeness for (fragments of) many-valued logics. In particular, various methods will be described for establishing structural completeness and its failure, and used to show for example that Gödel logic and product logic are structurally complete, while the implicational fragments of Łukasiewicz logic and basic logic are structurally complete, but not the full logics or indeed their implication-strong conjunction fragments. The historically pertinent case of the relevant logic RM (for which the disjunctive syllogism is admissible but not derivable) will be explored in detail and axiomatizations (bases) described for the admissible rules of various fragments.

References

[1] P. Cintula, G. Metcalfe, Structural completeness in fuzzy logics, Notre Dame Journal of Formal Logic 50 (2) (2009) 153–183.

[2] J. S. Olson, J. G. Raftery, C. J. V. Alten, Structural completeness in substructural logics, Logic Journal of the IGPL 16 (5) (2008) 453–495.

[3] V. Rybakov, Admissibility of Logical Inference Rules, Elsevier, 1997.

Toward a logic for imprecise probabilities

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Imprecise probabilities over fuzzy events are interpreted in terms of bets in de Finetti's style. That is, the upper probability of a fuzzy event ϕ is the the betting odd α that a bookmaker would accept for the following bet: the bettor pays α and gets the truth value of ϕ. The lower probability of ϕ is the amount β that a bookmaker would accept for the opposite bet (that is, the bookmaker pays β and gets the truth value of ϕ). Our coherence criterion (joint work with M. Fedel, K. Keimel and W. Roth) is given for a whole book (a finite system of pairs event-betting odd) and not for a single bet: a book is coherent if there is no bad bet, that is, there is no bet for which there is an alternative system of bets based on the book which gives the bettor a better payoff. A similar condition, but without direct reference to bad bets, is introduced in Walley's work on lower and upper previsions of gambles. In the first part of this talk, we compare Walley's approach with ours. In the second part of the talk we discuss a logical and algebraic approach to imprecise probabilities and to lower and upper previsions of gambles. Somewhat surprisingly, both topics can be treated inside a variety of universal algebras, consisting of MV-algebras with a unary operator reflecting the properties of upper probabilities.

Notes on the foundations of fuzzy logic

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Abstract

We discuss two approaches related to the foundational problem of fuzzy logic. The first one starts from a specific fuzzy logic and endows it with a non-standard semantics, with the intention that this semantics can be more easily understood than the canonical one. The second one starts from the application with which fuzzy logic traditionally claims to deal, namely reasoning about properties which are not clearly delimitable.

There has been a lot of discussion around the question on which principles t-norm based many-valued logics, or fuzzy logics for short, are based. A good amount of work deals with the interpretation of truth values (e.g., [HeCa]); further contributions try to justify particular choices of t-norms (e.g., [Law]); and a few approaches address the full framework of a fuzzy logical calculus (e.g., [Fer], [Par2]).

One line of research considers specific fuzzy logics and tries to endow them with alternative semantics (e.g., [Par1]). This amounts to the question if it is possible to make sense out of certain (classes of) residuated lattices. Ideal lattices [WaDi] gave originally rise to the notion, but seem to be unrelated to the present context. A more interesting approach can be found in [OnKo], where certain residuated lattices are represented in the power set of a suitable po-monoid, the monoidal operation being pointwise defined. This approach is particularly appealing if the underlying po-monoid is of a simple form, like for instance a Boolean algebra. Following these lines a semantics for Łukasiewicz logic can in fact be defined [Vet1]. A generalisation is even possible for Basic Logic, although with some loss of elegancy [Vet2].

Interpretations of this kind might open new fields of applications for fuzzy logics – logics whose mathematical theory is incredibly rich and fascinating. Still, the problem how fuzzy logics and their actually intended applications are precisely correlated remains a challenge. It seems to be well worth to change the perspective and to start from the side of intended applications.

I would like to mention a formalism which is apparently not in the scope of fuzzy logics: the logic of approximate reasoning [Rus, DPEGG]. The logic LAE [EGRV] interprets propositions as subsets of a metric space, and it allows to conclude from one proposition to another one even if the former is only approximately a subset of the latter. Namely, the statement $\alpha \overset{t}{\Rightarrow} \beta$ holds under some evaluation v if $v(\alpha) \subseteq U_t(v(\beta))$, where U_t assigns the t-neighborhood to some subset of the space.

According to a common understanding of fuzzy sets, a truth value measures the similarity to the nearest prototype (see, e.g., [DOP]). Indeed, a fuzzy set can, in the typical case, identified with its kernel and a pseudometric; the latter assigns to each point of the base set its distance to the kernel. This idea leads to a setting which is actually not far from the framework of LAE. We shall explore the relationship to a certain extent.

References

[DOP] D. Dubois, W. Ostasiewicz, H. Prade, Fuzzy sets: History and basic notions, in: D. Dubois et al. (eds.), Fundamentals of fuzzy sets, Kluwer Academic Publishers, Dordrecht 2000; 21 - 124.

[DPEGG] D. Dubois, H. Prade, F. Esteva, P. Garcia, L. Godo., A logical approach to interpolation based on similarity relations, *Int. J. Approx. Reasoning* **17** (1997), 1 - 36.

[EGRV] F. Esteva, L. Godo, R. O. Rodríguez, T. Vetterlein, On the logics of similarity-based approximate and strong entailment, Proc. of ESTYLF 2010; 187 - 192.

[Fer] C. G. Fermüller, Dialogue Games for Many-Valued Logics - an Overview. *Studia Logica* **90** (2008), 43 - 68.

[HeCa] H. M. Hersh, A. Caramazza, A fuzzy set approach to modifiers and vagueness in natural language, *Journal of Experimental Psychology: General* **105** (1975), 254 - 276.

[Law] J. Lawry, A voting mechanism for fuzzy logic, *Int. J. Approx. Reasoning* **19** (1998), 315 - 333.

[OnKo] H. Ono, Y. Komori, Logics without the contraction rule, *J. Symb. Log.* **50** (1985), 169 - 201.

[Par1] J. B. Paris, A semantics for fuzzy logic, *Soft Comp.* **1** (1997), 143 - 147.

[Par2] J. B. Paris, Semantics for fuzzy logic supporting truth functionality, in: V. Novák et al. (eds.), "Discovering the world with fuzzy logic", Physica-Verlag, 2000; 82 - 104.

[Rus] E. Ruspini, On the semantics of fuzzy logic, *International Journal of Approximate Reasoning* **5** (1991), 45 - 88.

[Vet1] Th. Vetterlein, Fuzzy logic as a logic of the expressive strength of information, *Soft Comput.* **12** (2008), 479 - 485.

[Vet2] Th. Vetterlein, A way to interpret Łukasiewicz Logic and Basic Logic, *Studia Logica* **90** (2008), 407 - 423.

[WaDi] M. Ward, R. P. Dilworth, Residuated lattices, *Trans. AMS* **45** (1939), 335 - 354.

Rough Sets and Many Valued Logics

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Introduction

Rough sets were defined almost 30 years ago by Z. Pawlak [Paw82] and since the begging attempts to relate them to three valued logic and algebraic semantic have been carried out. During the years, these kind of studies continued and still continue, involving also rough-set generalizations. Here we try to put in evidence the connections among rough sets, three-valued sets and algebraic logic. Let us recall the basic definitions of rough sets theory.

Definition 1. *An approximation space is a pair* (X, E) *with* X *a set of objects and* E *an equivalence (reflexive, symmetric, transitive) relation on* X*. Equivalence classes are denoted as* $[x]_E$*.*

On any approximation space it is possible to define the lower and upper approximation of a given set.

Definition 2. *Let* (X, E) *be an approximation space. The* lower approximation *of* $A \subseteq X$ *is* $l(A) := \{x \in X | [x]_E \subseteq A\}$ *and the* upper approximation *of* A *is* $u(A) := \{x \in X | [x]_E \cap A \neq \emptyset\}$*. A* rough set *is the pair lower-upper* $r(A) := (l(A), u(A))$ *or equivalently the pair lower-exterior* $r_e(A) := (l(A), e(A)) := (l(A), u^c(A))$*. A set* A *is said to be* exact *iff* $l(A) = A$ *or equivalently* $A = u(A)$*.*

Rough Sets and Three Values

Let f be a three-valued set on the universe X, $f : X \mapsto \{0, \frac{1}{2}, 1\}$. Then, from f we can induce three (Boolean) subsets of the universe:

$$A_1 := \{x : f(x) = 1\} \qquad \text{The positive domain}$$
$$A_0 := \{x : f(x) = 0\} \qquad \text{The negative domain}$$
$$A_u := \{x : f(x) = \tfrac{1}{2}\} \qquad \text{The neutral domain}$$

Thus, given the three-valued set f, we obtain the pair (A_1, A_0) satisfying the property $A_1 \cap A_0 = \emptyset$, i.e., A_1 and A_0 are disjoint. Vice versa, given a pair of disjoint sets, we can define a three-valued set in an obvious way: $f(x) = 1$ if $x \in A_1$; $f(x) = 0$ if $x \in A_0$ and $f(x) = \frac{1}{2}$ otherwise. Thus, we have a bijection between the collection of three-valued sets $\mathcal{F}_{\frac{1}{2}}(X) := \{f | f : X \mapsto \{0, \frac{1}{2}, 1\}\}$ and the collection of disjoint subsets (also called *orthopairs*) of X, $\mathcal{O}(X) := \{(A_1, A_0) | A_1, A_0 \in X; A_1 \cap A_0 = \emptyset\}$.
We note that from (A_1, A_0) another subset $A_p := A_0^c$ of the universe can be defined as the negation of the neutral domain. In some context, it is the *possibility domain*, collecting elements that possibly belong to a set . From the gradual truth perspective, A_1 is the *core* of the fuzzy set f and A_p its *support*. The collection of all nested pairs on X will be denoted as $\mathcal{N}(X) := \{(A, C) : A, C \subseteq X, A \subseteq C\}$. As can be seen a rough set $r(A)$ is made of a nested pair of sets (equivalently $r_e(A)$ is an orthopair). Thus, the collection $\mathcal{R}(X) = \{(L(A), U(A)) | A \subseteq X\}$ of all rough sets on a given universe X is a subset of $\mathcal{N}(X)$ (or equivalently $\mathcal{R}_e(X) = \{(L(A), U^c(A)) | A \subseteq X\} \subseteq \mathcal{O}(X)$).

From the above discussion, it can be seen that studying three-valued sets, we can infer properties on ortho (nested) pairs and then on rough sets. In particular, from operations on $\mathbf{3} = \{0, \frac{1}{2}, 1\}$ we immediately obtain operations on orthopairs. However difficulties may arise since rough sets are only a subset of all orthopairs. Thus, given an operation on $\mathbf{3}$, it must be shown that it is closed on $\mathcal{R}_e(X)$ and that the truth-functionality of the operation is preserved by the intended semantics. This is not always simple and even if from a theoretical point of view successful, it can pose problems on the interpretability of the results. This happens for instance for basic operations such as intersection and union. From a formal point of view we are able to define a lattice structure on $\mathcal{R}(X)$ (Section) but the meaning of these operations is not so straightforward.

Generalizations

There are several possible generalizations to the above definition of approximation space and rough sets. We can define the approximations using a general binary relation (not an equivalence one), we have probabilistic rough sets and fuzzy rough sets. As far as the link between rough set and many-valued logic is concerned, two generalization are worth to consider:

- Approximations of fuzzy sets. The underlying universe is no more made of by Boolean sets but by fuzzy sets which have to be approximated. This lead to the development of fuzzy rough sets and other approaches such as [CF08].
- There are situations where more than one approximation is available. If these approximations are "nested", i.e., they give rise to a so-called regular Approximation Framework, than we pass from 3 to n values [Ciu09]. These different approximations can arise for different reasons, for instance in case of multi-source (or multi-agent) information (an emerging field of investigation in rough sets theory).

Algebras of rough sets

The collections of ortho (nested) pairs and rough sets can be endowed with different operations and thus algebraic structures. We present an overview to the algebraic approach to orthopairs and rough sets.

Iwinski [Iwi87] studied the collection of orthopairs in connection with rough sets ideas and showed that $(\mathcal{N}(X), \sqcap, \sqcup, (\emptyset, X), (X, \emptyset))$ is a distributive lattice where the lattice operations are $(A_1, B_1) \sqcap (A_2, B_2) := (A_1 \cap A_2, B_1 \cup B_2)$ and $(A_1, B_1) \sqcup (A_2, B_2) := (A_1 \cup A_2, B_1 \cap B_2)$. This result can be improved in different ways:

- The structure $\langle \mathcal{O}(X), \sqcap, \sqcup, \approx, (\emptyset, X), (X, \emptyset)\rangle$ is a Stone algebra [PP88] where $\approx$ is defined as $\approx (A, B) := (B, B^c)$.
- The structure $\langle \mathcal{O}(X), \sqcap, \sqcup, \approx, -, (\emptyset, X), (X, \emptyset)\rangle$ is a BZ lattice [CN89] where $-(A, B) := (B, A)$.

Further, rough sets have been shown to be a model of several algebras. The stronger one is Heyting Wajsberg algebra [CCGK04a, CCGK04b] $\langle \mathcal{O}(X), \Rightarrow_L, \Rightarrow_G, (\emptyset, X)\rangle$ where $\Rightarrow_L, \Rightarrow_G$ are respectively a Łukasiewicz and a Gödel implication. We note that HW algebras are equivalent to other well know structures, for instance MV_Δ algebras. Worth to mention are also structures arising considering L,U as topological and modal operators. So for instance, in [BC04] rough algebras are obtained adding more axioms to topological quasi Boolean algebras and obtaining also representation theorems of these algebras wrt rough sets.

Finally, it is possible to define orthopairs on more abstract spaces.

- By defining orthopairs on a Boolean algebra A as the collection $\mathcal{C}(A) := \{(a, b) : a, b \in A, a \wedge b = 0\}$, Walker [Wal94] proved that the structure $\langle \mathcal{C}(A), \curlyvee, \curlywedge, \sim, (0, 1)\rangle$ is a Stone algebra, where $\curlyvee, \curlywedge$ are the abstract meet and join corresponding to the concrete ones $\sqcap, \sqcup$: $(a, b) \curlyvee (c, d) := (a \wedge b, c \vee d)$, $(a, b) \curlywedge (c, d) := (a \vee b, c \wedge d)$, $(0, 1)$ is the minimum of the lattice, $\sim$ is the corresponding of $\approx$: $\sim (a, b) := (b, b')$.
- The structure of nested pairs of a Boolean algebra, i.e., pairs (a, b) such that $a \leq b$, was studied also by Monteiro [Mon80, p.199]. Translating his results to orthopairs, he showed that $\langle \mathcal{C}(A), \curlyvee, \curlywedge, \neg, \nabla, (1, 0)\rangle$ where $\neg(a, b) := (b, a)$ and $\nabla(a, b) = (b', b)$ is a three-valued Łukasiewicz algebra.
- Considering an Heyting algebra $\langle A, \wedge, \vee, \rightarrow, 0, 1\rangle$, instead of a Boolean one, Vakarelov [Vak77] showed that $\langle \mathcal{C}(A), \curlyvee, \curlywedge, \twoheadrightarrow, \neg, \nabla, (1, 0)\rangle$ where $(a, b) \twoheadrightarrow (c, d) := (a \rightarrow c, a \wedge d)$ is a Nelson algebra.

References

[BC04] M. Banerjee and K. Chakraborty, *Algebras from rough sets*, Rough-Neural Computing (S.K Pal, A. Skowron, and L. Polkowski, eds.), Springer-Verlag, 2004, pp. 157–188.

[CCGK04a] G. Cattaneo, D. Ciucci, R. Giuntini, and M. Konig, *Algebraic structures related to many valued logical systems. part I: Heyting Wajsberg algebras*, Fundamenta Informaticae **63** (2004), no. 4, 331–355.

[CCGK04b] G. Cattaneo, D. Ciucci, R. Giuntini, and M. Konig, *Algebraic structures related to many valued logical systems. part II: Equivalence among some widespread structures*, Fundamenta Informaticae **63** (2004), no. 4, 357–373.

[CF08] D. Ciucci and T. Flaminio, *Generalized rough approximations in* $Ł\Pi\frac{1}{2}$, International Journal of Approximate Reasoning **48** (2008), 544–558.

[Ciu09] D. Ciucci, *Approximation algebra and framework*, Fundamenta Informaticae **94** (2009), no. 2, 147–161.

[CN89] G. Cattaneo and G. Nisticò, *Brouwer-Zadeh posets and three valued Łukasiewicz posets*, Fuzzy Sets and Systems **33** (1989), 165–190.

[Iwi87] T. Iwinski, *Algebras for rough sets*, Bulletin of the Polish Academy of Sciences, series: Mathematics **35** (1987), 673–683.

[Mon80] A. Monteiro, *Sur les algèbres de Heyting symétriques*, Portugaliae Mathematica **39** (1980), 1–237.

[Paw82] Z. Pawlak, *Rough sets*, Int. J. of Computer and Information Sciences **11** (1982), 341–356.

[PP88] J. Pomykala and J.A. Pomykala, *The stone algebra of rough sets*, Bulletin of the Polish Academy of Sciences - Mathematics **36** (1988), no. 7–8, 495–507.

[PS07] Z. Pawlak and A. Skowron, *Rudiments of rough sets*, Information Sciences **177** (2007), 3–27.

[Vak77] D. Vakarelov, *Notes on n-lattices and constructive logic with strong negation*, Studia Logica **36** (1977), 109–125.

[Wal94] E.A. Walker, *Stone algebras, conditional events, and three valued logic*, IEEE Transaction of Sysyems, Man, and Cybernetics **24** (1994), no. 12, 1699–1707.

Extended-order algebras and fuzzy implicators

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In the literature the term *fuzzy implicator* usually denotes a map $\mathcal{I} : [0,1] \times [0,1] \to [0,1]$ that satisfies the boundary conditions

$$(b)\ \mathcal{I}(0,0) = \mathcal{I}(0,1) = \mathcal{I}(1,1) = 1 \text{ and } \mathcal{I}(1,0) = 0.$$

Practically, it can be viewed as an extension of the implication defined in classical logic.
In different theoretical approaches and applications further properties are considered and assumed on $\mathcal{I}$. A detailed list of the most important is given in [3], where there is also a complete view of the interrelationships among these axioms.

In this work we reconsider the notion of implicator in a complete lattice L and discuss its properties, taking into account the viewpoint of the implication operation of *(weak) extended-order algebras*, defined in [2] and further developed in [1]. Such an implication can be viewed as an extension of an order relation, as it is explained in [2], and can be viewed as an implicator in L whose properties depend on those characterizing the structure of the algebra.

We also propose in a *(weak) right distributive complete extended-order algebra* $(L, \to, \top)$ with adjoint product $\otimes$ a *relative implication*

$$\to_\otimes : L \times L \to L, (a,b) \longmapsto a \to_\otimes b = a \to (a \otimes b),$$

as an alternative implicator beyond $\to$.
The implicators have been mostly considered to evaluate the inclusion between fuzzy sets, whose crisp version is defined, in the L-powerset L^X of any set X, as the order relation point-wisely induced by the lattice-order of L; in fact a non-crisp notion of inclusion between L-sets (subsethood degree) is usually defined point-wisely (either in "scalar" or "vector" form) by means of an implicator assigned in L, so extending the notion of crisp inclusion.
The relative implication allows a different extension of the inclusion relation between L-sets A and B that consists in seeing to which extent A is included in its conjunction with B.

Moreover, we introduce in $(L, \to, \top)$ the following binary operation that we call *conditional conjunction*

$$\otimes_\to : L \times L \to L, (a,b) \longmapsto a \otimes_\to b = a \otimes (a \to b),$$

that can be read as *"a"* and *"b, given a"*, which motivates the term we have chosen to denote it (note that in BL algebras this operation coincides with $\wedge$ but in most algebras we consider, such an equality does not hold).
This operation satisfies most conditions usually asked to a conjunction and it is well related to the adjoint product $\otimes$ and to the meet operation $\wedge$.

References

[1] M. E. Della Stella, C. Guido: *Associativity, commutativity and symmetry in residuated structures* (submitted).

[2] C. Guido, P. Toto: *Extended-order algebras*, Journal of Applied Logic, **6**(4) (2008), 609-626.

[3] Y. Shi, B. Van Gasse, D. Ruan, E. E. Kerre: *On dependencies and independencies of fuzzy implication axioms*, Fuzzy Sets and Systems (to appear).

A characterization of admissible Łukasiewicz assessments through t-SMV-algebras

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In [2], it is studied a betting situation (called *non-reversible betting game*) where the bookmaker fixes his betting odds on some fuzzy events and bettor can bet on them, but it is not allowed to interchange the roles of the players. In that paper, events are identified with MV-terms, or equivalence classes of formulas of Łukasiewicz logic, and the following result is proved:

Theorem 1. *Let $\phi_1, \ldots, \phi_n$ be events, and let $\mathbf{a} : \{\phi_1, \ldots, \phi_n\} \to [0,1]$ be a rational-valued assessment in a non-reversible betting game. Then the following are equivalent:*

(i) The assessment $\mathbf{a}$ *does not admit a* bad bet*, that is, a bet for which there is an alternative system of bets which guarantees a strictly better payoff, independently of the truth values of the events involved.*

(ii) There is a $t \leq n$, and a set of states $\{s_1, \ldots, s_t\}$ over the Lindenbaum algebra of Łukasiewicz logic generated by the propositional variables occurring in $\phi_1, \ldots, \phi_k$, such that, for every $i = 1, \ldots, n$,

$$\mathbf{a}(\phi_i) = \max\{s_j([\phi_i]) \mid j = 1, \ldots, t\}.$$

Let us call *admissible* a rational-valued Łukasiewicz assessment $\mathbf{a} : \{\phi_1, \ldots, \phi_n\} \to [0,1]$ in a non-reversible betting game that avoids bad bets and let us denote by Luk-Adm the set of all admissible assessments.

In order to investigate this betting games in an algebraic framework, we introduce the variety of t-SMV-algebras. These structures are a generalization of SMV-algebras (cf. [3]) and allow us to treat and characterize admissible assessments in terms of satisfiability of suited defined equations in their language. In [1], the authors prove that the problem of deciding satisfiable equations in SMV-algebra is NP-complete. Here we use a similar technique to show that the problem to check satisfiability of equations in t-SMV-algebras is also NP-complete. As a consequence of this result, we obtain that Luk-Adm is NP-complete.

Definition 1. *For every $t \geq 1$ a* t-SMV*-algebra is an algebra $\mathcal{A} = (A, \sigma_1, \ldots, \sigma_t)$ where A is an MV-algebra and, for every $i, k, h = 1, \ldots, t$, the following equations are satisfied:*

(i) $\sigma_i(\bot) = \bot$

(ii) $\sigma_i(\neg x) = \neg \sigma_i(x)$

(iii) $\sigma_i(\sigma_k(x) \oplus \sigma_h(y)) = \sigma_k(x) \oplus \sigma_h(y)$

(iv) $\sigma_i(x \oplus y) = \sigma_i(x) \oplus \sigma_i(y \ominus (x \odot y))$

Clearly the class of t-SMV-algebras forms a variety we will denote by $\mathbb{TSMV}$.

Theorem 2. *Let $\phi_1, \ldots, \phi_t$, be* t-SMV*-terms in k variables and let $\mathbf{a} : \phi_i \mapsto k_i/z_i$ (for $i = 1, \ldots, t$) be a rational-valued Łukasiewicz assessment. Then, the following are equivalent:*

(i) $\mathbf{a}$ *is admissible.*

(ii) There exists a t-SMV*-algebra $(A, \sigma_1, \ldots, \sigma_t)$ satisfying, for all $i = 1, \ldots, t$, the equations ε_i: $(z_i - 1)x_i = \neg x_i$ and δ_i: $k_i x_i = \left(\bigvee_{j=1}^{t} \sigma_j(\phi_i)\right)$, where the variables x_i's are fresh.*

In other words the admissibility of $\mathbf{a}$ is witnessed by the satisfiability, in the class of t-SMV-algebras, of the set of equations

$$\Phi = \{\varepsilon_i, \delta_i : i = 1, \ldots, t\}. \quad (1)$$

Then we study the computational complexity for the problem of checking satisfiability of equations in t-SMV-algebras and we prove the following.

Theorem 3. *The problem of checking the satisfiability of an equation in a* t-SMV*-algebra is NP-complete.*

As a consequence, we have an algorithm for checking admissibility of an assessment. Given a rational-valued Łukasiewicz assessment $\mathbf{a} : \phi_i \mapsto k_i/z_i$ (with $i = 1, \dots, t$), we do the following:

(i) For all $i = 1, \dots, t$, define the terms ε_i and δ_i as in Theorem 2.

(ii) Define the set of equations Φ as in (1).

(iii) Check the satisfiability of Φ in the class of t-SMV-algebras.

Being the binary encoding of Φ polynomial in t, the above algorithm joint with Theorem 3 ensures that:

Lemma 1. Luk-Adm *is in NP.*

NP-hardness of Luk-Adm immediately follows from the NP-hardness of satisfiability of MV-equations. Therefore:

Theorem 4. *Deciding* Luk-Adm *is NP-complete.*

References

[1] S. Bova, T. Flaminio: The coherence of Łukasiewicz assessment is NP-complete. International Journal of Approximate Reasoning 51, 294–304, 2010.

[2] M. Fedel, K. Keimel, F. Montagna, W. Roth.: Imprecise probabilities, bets and functional analytic methods in Łukasiewicz logic. Submitted.

[3] T. Flaminio, F. Montagna: MV-algebras with Internal States and Probabilistic Fuzzy Logics. International Journal of Approximate Reasoning 50, 138–152, 2009.

MV-semirings and their sheaf representation

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In this paper we present a categorical equivalence between MV-algebras and MV-semirings. The notion of MV-semirings were introduced by A. Di Nola and B. Gerla in [3] as a coupled semiring. In [1], the authors gave an alternative and equivalent definition of MV-semirings, that we use in our paper.

Definition 1. *Let $S = (S, +, \cdot, 0, 1)$ be a commutative additively idempotent semiring. We call S a MV-semiring iff for each element $s \in S$, there exists the residuum with respect to 0, i.e. there exists a greatest element s^* such that $s \cdot s^* = 0$ and such that $s + t = (s^* \cdot (s^* \cdot t)^*)^*$, for each $s, t \in S$.*

Theorem 1. *The category of MV-algebras is categorical equivalent with respect to the category of MV-semirings.*

This approach allows to keep the inspiration and use new tools from semiring theory to analyze the class of MV-algebras.

The situation of ideals in an MV-semiring requires clarification. This is so for a couple of reasons. One is that given an MV-algebra A and its extracted MV-semiring S, the latter inherits some of its ideals from A. But these ideals sit in S differently than they sit in A. For example an ideal may be prime in A while in S it's not prime. Despite this, there is a link between the prime spectrum of an MV-semiring and the prime spectrum of an MV-algebra given by

Theorem 2. *Let A be an MV-algebra and S the MV-semiring associated with A in the categorical equivalence. Consider* $\mathrm{Spec}(A)$ *topologized with the CoZariski topology and* $\mathrm{Spec}(S)$ *with the Zariski topology. It results that* $\mathrm{Spec}(A)$ *and* $\mathrm{Spec}(S)$ *are homeomorphic.*

In [2], Chermnykh gives a sheaf representation for commutative semirings in analogy to the sheaf representation given by Grothendieck for rings. We specialize this representation for MV-semirings. The main tool for such a representation is given by the localization of MV-semirings over prime ideals. Althought these localizations are not MV-semirings, we still have a representation theorem for MV-semirings in terms of sections of sheaves that can be easily translated in an MV-algebraic fashion.

Let S be a commutative idempotent semiring with unit and $D \subseteq S \setminus \{0\}$ a multiplicative monoid, i.e. $1 \in D$ and D is closed under $\cdot$. From S we can construct a semiring in the following way.

Let $(a, b),\ (c, d) \in S \times D$ and define $(a, b) \sim (c, d)$ if and only if there exists an element $k \in D$ such that $adk = bck$. It is easy to verify that $\sim$ is an equivalence relation. In the follow we denote by S_D the quotient of $S \times D$ by $\sim$ and by a/b the equivalence class of the pair (a, b). It results that S_D is a semiring with the following operations:

$$a/b + c/d = (ad + bc)/bd,$$
$$a/b \cdot c/d = ac/bd.$$

The $\mathbf{0}$ is the class $0/1$ and the $\mathbf{1}$ is the class $1/1$.

Proposition 1. *Let $(S, +, \cdot, 0, 1)$ be a commutative additively idempotent semiring and $D \subseteq S \setminus \{0\}$ a multiplicative monoid. Then $(S_D, +, \cdot, \mathbf{0}, \mathbf{1})$ is also a commutative additively idempotent semiring.*

Now let $P \in \mathrm{Spec}(S)$ and set $D = S \setminus P$. D is a multiplicative monoid and S_D is a local semiring (see [2]). We write S_P for S_D in this case and S_P is named *the localization of S at P.*

If S is an MV-semiring and $P \in \mathrm{Spec}(S_r)$ in general S_P is not an MV-semiring. Consider the following example.

Example 1. *Let $S = \langle C,\ +,\ \cdot,\ 0,\ 1\rangle$ be the Chang MV-semiring, i.e., C is the Chang algebra. Let M be the radical of C. Then M is prime in S, so $D = S \setminus M = \{(nc)^* \mid n \geq -\}$, c the atom of C. If S_M were an MV-semiring we would have for $n > 0$, $1/(nc)^* + 1/1 = 1/1$, so $(1 + (nc)^*)/(nc)^* = 1/1$. But $1 + (nc)^* = 1$ so we would have $1/(nc)^* = 1/1$. Thus for some $w \in D$ we have $w = (nc)^* w$. Now $w = (mc)^*$ for some $m \geq 0$. So we obtain $(mc)^* = (mc)^*(nc)^* = ((m+n)c)^*$. Hence $mc = (m+n)c$ and this implies $m + n = m$ so $n = 0$ contrary to assumption.*

Indeed, it results that

Proposition 2. *Let S be an MV-semiring and $P \in \mathrm{Spec}(S_r)$. S_P is a commutative additively idempotent local semiring.*

Let S be a commutative semiring, the Grothendieck sheaf of S is the triple $G(S) = (\mathrm{Spec}(S), E_S, \pi_S)$ where $E_S = \bigcup\{S_P \times \{P\} : P \in \mathrm{Spec}(S)\}$ and $\pi : E_S \to \mathrm{Spec}(S)$, defined as $\pi(a/b, P) = P$, is a local homeomorphism. In the follow, we denote by $[s/t]_P$ the element $(s/t, P) \in E_S$ and by $(\widehat{S}, \widehat{+}, \widehat{\cdot}, \widehat{0}, \widehat{1})$ the semiring of all global sections i.e. of the continuous maps of type $\widehat{s} \mid \mathrm{Spec}(S) \to E_S$ such that $\widehat{s}(P) = [s/1]_P \in S_P$, where

$$\begin{aligned}(\hat{s}\widehat{+}\hat{t})(P) &= \hat{s}(P) + \hat{t}(P) = \widehat{s+t}(P)\\ (\hat{s}\hat{\cdot}\hat{t})(P) &= \hat{s}(P) \cdot \hat{t}(P) = \widehat{s \cdot t}(P)\end{aligned}$$

and

$$\begin{aligned}\hat{0} : P \in \mathrm{Spec}(S) &\to 0_P \in E_S\\ \hat{1} : P \in \mathrm{Spec}(S) &\to 1_P \in E_S\end{aligned}$$

From the results of Chermnykh in [2], S and $\widehat{S}$ are isomorphic as semirings. But, in the case of MV-semirings it results that $\widehat{S}$ is also an MV-semiring.

Theorem 3. *An MV-semiring is isomorphic to the MV-semiring of all global sections of its Grothendieck sheaf.*

This representation can be traslated in terms of MV-algebras.
Let A be an MV-algebra and $\Delta(A)$ the MV-semiring associated. By Theorem 3, $\Delta(A)$ is isomorphic to the MV-semiring $\widehat{\Delta(A)} = \{\hat{a} : \mathrm{Spec}(\Delta(A)) \to E_{\Delta(A)} \mid \hat{a}$ is continuous and $\hat{a}(P) \in \Delta(A)_P$, for each $P \in \mathrm{Spec}(\Delta(A)_r)\}$.

Theorem 4. *Each MV-algebra A is isomorphic to the MV-algebra of all global sections of the Grothendieck sheaf of the reduct semiring associated with A.*

It is worth stressing again, that in our representation the stalks are not MV-semirings but only commutative additively idempotent semirings. Despite that, the algebra of all global sections in the sheaf representation is still an MV-semiring.

References

[1] Belluce L. P., DiNola A., *Commutative rings whose ideals form an MV-algebra*, MLQ Math. Log. Q. **55** (2009) no. 5, p. 468-486.

[2] Chermnykh V. V., *Sheaf representations of semirings*, Russian Math. Surveys **5** (1993), p. 169-170.

[3] Di Nola A., Gerla B., *Algebras of Lukasiewicz's Logic and their semiring reducts*, Contemporary Mathematics **377** (2005), p. 131-144.

Towards a betting interpretation for necessity measures of many-valued events

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The aim of this work is to make a first step in the study of a betting interpretation in the style of de Finetti for necessity measures over many-valued events. In particular, we are interested in dealing with necessity measures over MV-algebras of functions $\mathcal{F}(X)$ of the form $\mathcal{F}(X) = \langle \mathcal{F}(X), \oplus, \neg, \mathbf{0}, \mathbf{1} \rangle$, where $\mathcal{F}(X)$ is the set of all functions from a finite set X into $[0, 1]$, $\oplus$ and $\neg$ result from the pointwise application of the real functions $x \oplus y = \min\{1, x + y\})$ and $\neg x = 1 - x$, and $\mathbf{1}$ and $\mathbf{0}$ are constant functions.

A betting interpretation in the style of de Finetti for probability measures over MV-algebras in terms of states has already been deeply studied. Denote by $\mathcal{H}$ the set of all the homomorphisms from $\mathcal{F}(X)$ into the standard MV-algebra $[0, 1]$ (cf. [1]).

Definition 1 ([6]). *A map $s : \mathcal{F}(X) \to [0, 1]$ is said to be a* state *if:*

(s1) $s(\mathbf{1}) = 1$,

(s2) whenever $\neg(\neg f \oplus \neg g) = \mathbf{0}$, $s(f \oplus g) = s(f) + s(g)$.

Let $\{f_1, \ldots, f_m\}$ be a finite subset of $\mathcal{F}(X)$, and let

$$\mathbf{a} : f_i \mapsto \beta_i \in [0, 1], \text{ for } i = 1, \ldots, m \tag{2}$$

be a mapping. We say that $\mathbf{a}$ satisfies *de Finetti coherence criterion* ([2]) iff for every $\sigma_1, \ldots, \sigma_m \in \mathbb{R}$, there is a $\mathbf{h} \in \mathcal{H}$ such that $\sum_{i=1}^{m} \sigma_i(\mathbf{a}(f_i) - \mathbf{h}(f_i)) \geq 0$.

The following result generalizes de Finetti's Theorem to the case of states over MV-algebras.

Theorem 1 ([7, 5]). *Let $\mathcal{F}(X)$, $f_1, \ldots, f_m$, and $\mathbf{a}$ be as above. Then the following are equivalent:*

(i) $\mathbf{a}$ *satisfies de Finetti coherence criterion.*

(ii) The map $\mathbf{a}$ *extends to a state on* $\mathcal{F}(X)$.

(iii) There is a probability measure μ on $\mathcal{H}$ such that for every $i = 1, \ldots, m$,

$$\mathbf{a}(f_i) = \int_{\mathcal{H}} \mathbf{h}(f_i) d\mu.$$

(iv) $\mathbf{a}$ *extends to a convex combination of elements in* $\mathcal{H}$.

The aim of this work is to find a suitable notion of coherence and a suitable betting interpretation, inspired by the above characterization, for necessity measures over MV-algebras of functions $\mathcal{F}(X)$. There are several notions of necessity measures over MV-algebras of functions (fuzzy sets). In this work we consider the following definition.

Definition 2 ([4]). *A map $N : \mathcal{F}(X) \to [0, 1]$ is said to be a* necessity measure *if the following conditions are satisfied:*

(N1) $N(\mathbf{1}) = 1$ *and* $N(\mathbf{0}) = 0$.

(N2) $N(f \wedge g) = \min(N(f), N(g))$.

(N3) for every real number r, denote by $\mathbf{r}$ the function constantly equal to r. Then for every $r \in [0, 1]$, $N(\mathbf{r} \oplus f) = r \oplus N(f)$.

Let $\pi : X \to [0,1]$ be normalized possibility distribution (i.e. $\max_{x\in X} \pi(x) = 1$). Then we define the *generalized Sugeno integral given by* π, as the map $\oint \cdot d\pi : \mathcal{F}(X) \to [0,1]$ such that, for every $f \in \mathcal{F}(X)$:

$$\oint f d\pi = \max_{x\in X}(\pi(x) \otimes f(x)).$$

where $v \otimes u = 1 - ((1-u) \oplus (1-v))$. We are interested in answering the following question:

Can we find a characterization of necessity measures in the style of Theorem 1?

In order to find such a characterization, we need to borrow some concepts and techniques from Tropical Mathematics (and tropical algebraic geometry). Tropical mathematics is a rapidly growing area of modern mathematics that investigates the properties of the mathematical structure of the reals $\mathbb{R}$ that arises when we replace in $\langle \mathbb{R}, +, \cdot, 0, 1\rangle$ the product by the sum, and the sum by a idempotent operation, usually the *minimum* $\wedge$, or the *maximum* $\vee$. The structure $\mathbf{R}_\wedge = \langle \mathbb{R}, \min, +\rangle$ is called the min-plus semiring.

We are particularly interested in the notion of tropical convexity, whose study has quite a long tradition, and goes back to the earliest work of Vorobyev [8] and Zimmerman [9]. Take the tropical min-plus semiring $\mathbf{R}_\wedge$ and extend $+$ and min to any $\mathbb{R}^n$ by the usual componentwise application of the operations in $\mathbf{R}_\wedge$. In an analogous way, if $\lambda \in \mathbb{R}$, and $x = (x(1), \ldots, x(n)) \in \mathbb{R}^n$, for $\star$ being any operation in $\mathbf{R}_\wedge$, we write $\lambda \star x$ instead of $(\lambda \star x(1), \ldots, \lambda \star x(n))$.

Fix a finite set $V = \{v_1, \ldots, v_k\}$ of points in $\mathbb{R}^n$. Then, $x \in \mathbb{R}^n$ is called to be:

(i) a *tropical convex combination* of V iff there are $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that

$$x = \bigwedge_{i=1}^{k} \lambda_i + v_i.$$

We denote by tconv(V) the set of points in $\mathbb{R}^n$ that are tropical convex combinations of V.

(ii) a *bounded normalized tropical convex combination* of V iff there are $\lambda_1, \ldots, \lambda_k \in \mathbb{R}^+$ such that $\bigvee_{i=1}^{k} \lambda_i = 1$, and

$$x = \bigwedge_{i=1}^{k} (1 - \lambda_i) \oplus v_i.$$

We denote by bn-tconv(V) the set of points in $\mathbb{R}^n$ that are bounded normalized tropical convex combinations of V.

Now, results in [4] allows us to characterize those assessments $\mathbf{a}$ defined on a finite subset of $\mathcal{F}(X)$ that extend to necessity measures.

Theorem 2. *Let* $\{f_1, \ldots, f_m\} \subseteq \mathcal{F}(X)$*, and let* $\mathbf{a}$ *be a mapping from* $\{f_1, \ldots, f_m\}$ *into* $[0,1]$*. Then the following are equivalent:*

(i) The map $\mathbf{a}$ *extends to a necessity measure on* $\mathcal{F}(X)$*.*

(ii) There exists a normalized distribution $\pi : X \to [0,1]$*, such that for every* $i = 1, \ldots, m$*,*

$$\mathbf{a}(f_i) = 1 - \oint_X \neg f_i \, d\pi.$$

(iii) $\langle \beta_1, \ldots, \beta_m\rangle \in$ *bn-tconv*$(\{\langle f_1(x), \ldots, f_m(x)\rangle : x \in X\})$*.*

Theorem 2 offers a characterization of the extension of necessity measures in terms generalized Sugeno integrals, and in terms of convex combinations. However, a coherence criterion in the style of de Finetti is still lacking. We plan to tackle this issue in our future work, and also to investigate extensions of this approach to different ways to define necessity and possibility measures over MV-algebras (see [4]).

References

[1] R. Cignoli, I.M.L. D'Ottaviano, D. Mundici, Algebraic Foundations of Many-valued Reasoning. Kluwer, Dordrecht, 2000.

[2] B. de Finetti, Sul significato soggettivo della probabilità , Fundamenta Mathematicae 17 (1931) 298–329. Translated into English as "On the subjective meaning of probability", in: Paola Monari and Daniela Cocchi (Eds.), Probabilità e Induzione, Clueb, Bologna, pp. 291–321, 1993.

[3] M. Develin, B. Strumfels, Tropical Convexity. Documenta Mathematica 9 (2004) 1–27.

[4] T. Flaminio, L. Godo, E. Marchioni, On the Logical Formalization of Possibilistic Counterparts of States over n-valued Łukasiewicz Events. Journal of Logic and Computation. doi:10.1093/logcom/exp012.

[5] J. Kühr, D. Mundici, De Finetti theorem and Borel states in $[0, 1]$-valued algebraic logic. International Journal of Approximate Reasoning 46(3), 605–616, 2007

[6] D. Mundici, Averaging the truth-value in Łukasiewicz logic. Studia Logica, 55(1), 113–127, 1995.

[7] D. Mundici, Bookmaking over Infinite-valued Events. International Journal of Approximate Reasoning 43, 223–240, 2006.

[8] N. N. Vorobyev, Extremal algebra of positive matrices, Elektron. Informationsverarb. und Kybernetik 3 (1967), 39-71, (in Russian).

[9] K. Zimmermann, A general separation theorem in extremal algebras, Ekonom.-Mat. Obzor 13 (1977), no. 2, 179-201. MR 56 #11869

Characterization of functions among lattice-valued relations

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Since the inception of fuzzy set theory and, more generally, of lattice-valued mathematics, through the big development of many valued logic there have been several attempts to present and study suitable generalizations of the concept of function within the universe of many-valued relations.
It is not the purpose of this talk to discuss and compare those generalizations and to present a list of them, even; we only note that, within some lines of the approach to the topic, an important role is played by the powerset operators, detemined by lattice-valued relations, that allow the transport of lattice-valued sets, which are related by suitable Galois connections and, in particular, adjunctions.
These operators and their properties for binary relations with values in a complete commutative residuated lattice L are described in [1], where $\top$-functional and $\bot$-functional L-relations are also considered as generalizations of functions.

However, a characterization of those lattice-valued relations that are functions has never been found or even asked. Here we give such a characterization that is surprisingly simple, since it reads just like the well known analogous in the classical case. In fact we prove that, for quite general lattice-ordered algebras L of many-valued logics, (crisp) functions are exactly those L-valued relations whose "existential" and "universal" backward powerset operators coincide.

References

[1] A. Frascella, C. Guido, *Transporting many valued sets along many valued relations*, Fuzzy Sets and Systems, **159**(1) (2008), 1-22.

Modelling degrees of truth: A case for objective probability

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Łukasiewicz is the chief perpetrator of a magnificent conceptual deceit lasting out in mathematical logic to the present day.
[9]

Chris Fermüller has recently put forward a convincing interpretation of Łukasiewicz real valued logic based on Giles's games [see, eg. 4, 3]. As the latter are based on probability, this solution can be seen as muddling up the clear and widely endorsed distinction between degrees of truth and degrees of belief. I argue that the distinction need not be given up, provided that we allow for a suitably interpreted notion of objective probability.

Background

There are two traditions which aim at extending classical logic to model reasoning under imperfect information. One extends the classical notion of logical consequence by dropping the requirement that it should only lead to certain conclusions. The other rejects the idea that the semantics underlying logical reasoning should be two-valued.

The former, which is currently well represented by the *Objective Bayesianism* programme [see, e.g. 8, 7, 10] focusses primarily on the concept of *degrees of belief*, which are formally represented by probabilities (whence Bayesianism). Unlike non logical probability, where we can trace its roots, this tradition takes the probabilistic representation of belief as the starting point to construct adequate *extensions* of classical logic (whence Objective).

The latter tradition is being actively pursued within the *Mathematical Fuzzy Logic* programme [see, eg 5, 6], whose primary goal consists in extending two-valued logics in order to model reasoning under vagueness (whence Fuzzy). This allows for a formal representation, mainly through suitable algebraic structures [1], of *degrees of truth*, a tradition which can be traced back to the development of many-valued logics in the 1920's.

From a formal point of view, probability and many-valued logics, share a good deal of mathematical structure, a fact which eventually depends on the properties of the real unit interval. This led de Finetti to the rather enthusiastic remark that

> There is no possible doubt, after the beautiful research of Łukasiewicz, Reichenbach, Mazurkiewicz et al., that the calculus of probability can be considered as a many-valued logic (precisely: a continuous scale of values), and that this point of view is the best one for elucidating the fundamental concept and logic of probability. [2]

Yet, there is wide consensus on the idea that probabilistic logic and many-valued logics model two *distinct* objects: degrees of belief and degrees of truth, respectively. The distinction has been vigorously advocated by scholars of both traditions and today it is generally taken for granted. The purpose of this note is to review some key aspects of the distinction and show how a difficulty which arises in connection with Giles's semantics for Łukasiewicz logic may shed new, interesting light, on the notion of *objective probability.*

A conceptual quandary

The distinction between 'degrees of belief' and 'degrees of truth' is not just part of the folklore. Indeed it can be argued for both intuitively and formally.

Intuitively, one can be very sure and absolutely wrong at the same time. Othello's belief about Desdemona's handkerchief makes a clear case in point. Conversely, one could be right, but without good reasons, as Gettier reminded us in connection to the justification of belief. What seems to be at the bottom of this intuitive distinction is that truth appears to be grounded on objective states of affairs, whereas belief essentially amounts to an agent's subjective disposition or attitude towards those states of affairs. I will argue, however, that we cannot really push this subjective *vs.* objective argument beyond the merely anecdotal.

Formally we can reason as follows. If we endorse the Bayesian point of view, the Dutch Book Theorem gives us a formal equivalence between rational degrees of belief and probability values. The right-to-left direction of the equivalence grants us the conclusion that anything which satisfies the laws of probability counts as rational assignment of degrees of belief. The latter are then clearly being separated from degrees of truth, which, among other things, are usually thought of as being compositional.

Putting this together it follows that we can't model (i.e give semantics to) degrees of truth via *subjective* probability. For if we did, we would really be representing degrees of belief. Giles's style semantics for Łukasiewicz logic seems to put us in a sort of epistemological quandary, unless, that is, we admit that the probability involved in this semantics is *objective.* This paper explores such a possibility and some of its consequences.

One apparently welcome consequence of this interpretation relates to the well known difficulties connected with the logical characterization of degrees of truth. As Suszko put it, 'Obviously, any multiplication of logical values is a mad idea' [9]. Indeed his point of view is substantiated by Wójcicki's result to the effect that any n-valued consequence relation is logically two-valued. As a consequence, the possibility of there being a logical notion of many-valuedness is altogether ruled out, a position which is usually referred to as Susko's Thesis. Intriguingly enough, as we shall see, de Finetti had very similar remarks about logical probability.

References

[1] R. Cignoli, I.M.L. D'Ottaviano, and D. Mundici. *Algebraic Foundations of Many-Valued Reasoning.* Kluwer Academic Publishers, 1999.

[2] Bruno de Finetti. The Logic of Probability (1935). *Philosophical Studies*, 77:181–190, 1995.

[3] Christian Fermüller. On Giles style dialogue games and hypersequent systems. In H. Hosni and F. Montagna, editors, *Probability, Uncertainty and Rationality*, pages 169–196. Edizioni della Normale, 2010.

[4] Christian Fermüller and George Metcalfe. Giles's Game and the Proof Theory of Łukasiewicz Logic. *Studia Logica*, 92(1):27–61, 2009. ISSN 0039-3215.

[5] Petr Hájek. *Metamathematics of Fuzzy Logic.* Kluwer Academic Publishers, 1998.

[6] Petr Hájek. On Vagueness, Truth Values and Fuzzy Logics. *Studia Logica*, 91(3):367–382, 2009.

[7] E.T. Jaynes. *Probability theory: the logic of science.* Cambridge University Press, 2003.

[8] J. B. Paris. *The uncertain reasoner's companion: A mathematical perspective.* Cambridge University Press, 1994.

[9] Roman Suszko. The Fregean Axiom and Polish mathematical logic in the 1920s. *Studia Logica*, 36(4): 377–380, 1977.

[10] Jon Williamson. *In Defence of Objective Bayesianism.* Oxford University Press, 2010.

Central limit theorem on MV-algebras

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The aim is to approve the Central limit theorem on MV-algebras by the new approach, using the observable as a distribution function, and not a σ-homomorphism. The main idea is in local representation of σ-algebras.

The following theorem is proved: Let M be a σ-complete MV-algebra with product, $m : M \longrightarrow [0,1]$ be a σ- additive state, $(x_n)_n$ be a sequence of independent, equally distributed, square integrable observables, $E[x_1] = E[x_2] = ... = a$, $\sigma(x_1) = \sigma(x_2) = ... = \sigma$. Then for any t $\in R$

$$\lim_{n\to\infty} m\left(\frac{\frac{1}{n}\sum_{i=1}^{n} x_i - a}{\frac{\sigma}{\sqrt{n}}}((-\infty,t))\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{u^2}{2}}du$$

1. MV-algebras. Definition. An MV-algebra is an algebraic system

$$(M, \oplus, \odot, \leq, 0, u),$$

where

$$M = [0,u]$$

is an interval in an l-group $G = (G, +, \leq), 0$ is the neutral element of G (i.e. $a + 0 = a$ for any $a \in G$), u is the strong unit of G (i.e. to any $a \in G$ there exists $n \in N$ such that $a \leq u + u + ... + u(n\text{-times})$),
$a \oplus b = (a + b) \wedge u$,
$a \odot b = (a + b - u) \vee 0$.

2. Definition. A state on an MV-algebra M is a mapping $m : M \to [0,1]$ satisfying the following conditions:
(i) $m(u) = 1, m(0) = 0$;
(ii) $a_n \nearrow a \Longrightarrow m(a_n) \nearrow m(a)$;
(iii) $a_n \searrow a \Longrightarrow m(a_n) \searrow m(a)$.

3. Definition. Let $\mathcal{J} = \{(-\infty, t); t \in R\}$. An observable on M is any mapping $x : \mathcal{J} \to M$ satisfying the conditions:
(i) $t_n \nearrow \infty \Longrightarrow x((-\infty, t_n)) \nearrow u$;
(ii) $t_n \searrow -\infty \Longrightarrow x((-\infty, t_n)) \searrow 0$;
(iii) $t_n \nearrow t \Longrightarrow x((-\infty, t_n)) \nearrow x((-\infty, t))$.

4. Theorem. Let $m : M \to [0,1]$ be a state, $x : \mathcal{J} \to M$ be an observable. Define $F : R \to [0,1]$ by the formula
$F(t) = m(x((-\infty, t))), t \in R$, then $F : R \to [0,1]$ is the distribution function.

5. Definition. An observable $x : \mathcal{J} \to M$ is called to be integrable if there exists

$$E(x) = \int_R t dF(t) = \sigma^2(x) + E(x)^2$$

where $F : R \to [0,1]$ is the distribution function of the observable x. The observable x is square integrable, if there exists

$$\int_R t^2 dF(t).$$

6. MV-algebra with product. An MV-algebra with product is a pair$(M, .)$, where M is an MV-algebra and . is a commutative and associative binary operation on M, and $u \,.\, a = a$ for any $a \in M$; $a \,.((b - c) \vee 0) = (a.b - a.c) \vee 0$ for any $a, b, c \in M$.

7. Definition Let M be a σ-complete MV-algebra with product, $x, y : \mathcal{J} \longrightarrow M$ be observables. Then its sum is defined by the formula

$$(x + y)\,((-\infty, t)) = h_n\left(g^{-1}\,((-\infty, t))\right).$$

8. Theorem. Generally,for n summands, we have: the sum of observables $x_1, ..., x_n : \mathcal{J} \longrightarrow M$ is the mapping

$$\left(\sum_{i=1}^{n} x_i\right)((-\infty, t)) = h_n\left(g^{-1}\,((-\infty, t))\right).$$

$$\sum_{i=1}^{n} x_i = h_n \circ g_n^{-1}.$$

9. Definition. Observables $x_1, ..., x_n$ are independent, if for any $t_1, ..., t_n \in R$

$$m\,(h_n\,((-\infty, t_1) \times (-\infty, t_2) \times ... \times (-\infty, t_n))) =$$

$$= m\,(x_1\,((-\infty, t_1)))\,.\,m\,(x_2\,((-\infty, t_2)))\,.\, ... \,.m\,(x_n\,((-\infty, t_n)))\,.$$

10. Definition. An observable $x : \mathcal{J} \longrightarrow M$ is called strong, if

$$[a, b) \cap [c, d) = \emptyset \Longrightarrow (x\,([a, b))\,.\alpha) \wedge (x\,([c, d))\,.\beta) = 0$$

for any $\alpha, \beta \in M$.

11. Definition. A state $m : M \longrightarrow \langle 0, 1\rangle$ is called σ-additive, if

$$m\left(\bigvee_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m\,(A_n)$$

whenever $A_n \cap A_m = \emptyset \ (n \neq m)$.

12. Central limit theorem Let M be a σ-complete MV-algebra with product, $m : M \to [0, 1]$ be a σ-additive state, $(x_n)_n$ be a sequence of independent, equally distributed, square integrable observables,$E\,[x_1] = E\,[x_2] = ... = a$, $\sigma(x_1) = \sigma(x_2) = ... = \sigma$. Then for any t $\in R$

$$\lim_{n\to\infty} m\left(\frac{\frac{1}{n}\sum_{i=1}^{n} x_i - a}{\frac{\sigma}{\sqrt{n}}}\,((-\infty, t))\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{u^2}{2}}\,du$$

hence $\xi_k : R^N \to R$ and $x_k : \mathcal{J} \to M$ have the same distribution function, and

$$E(\xi_k) = \int_R t dF_k(t) = E(x_k).$$

References

[1] Riečan, B.: On a new approach to probability theory on MV-algebras (to appear).

[2] Riečan, B., Lašová, L.: On the probability theory on the Kôpka D-posets (to appear).

[3] Montagna, F.: An algebraic approach to propositional fuzzy logic. J.Logic Lang.Inf. 2000, 91-124.

[4] Riečan, B., Mundici, D.: Probability on MV algebras, Handbook of Measure Theory, Elsevier, Amsterdam 2002, 869 - 909.

[5] Riečan, B.: On the product MV-algebras. Tatra Mt. Math. Publ. 16, 1999, 143 - 149.

Individual ergodic theorem on Kôpka D-posets

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D-posets include many algebraic structures, which are known in quantum mechanic and in fuzzy sets theory. The definition of D-poset was introduce by Chovanec and Kôpka in the work [1]. In this paper we work with probability on the special type of D-poset and we formulate and prove the Individual ergodic theorem on D-posets. As first we define the structure which is called D-poset. This definition we can find in work [6].

Definition 1. *The system $\mathcal{D} = (D, \leq, -, 0_D, 1_D)$ is called D-poset, if $(D, \leq)$ is partially ordered set with the smallest element 0_D and the largest element 1_D, The partially binary operation $-$ is defined on the set D and the following statements hold for each element $a, b, c \in D$:*

(i) $b - a$ *is defined if and only if* $a \leq b$;

(ii) $\forall a \leq b \in D$: $b - a \leq b$ *and* $b - (b - a) = a$;

(iii) if $a \leq b \leq c$, *then* $c - b \leq c - a$ *and* $(c - a) - (c - b) = b - a$.

Definition 2. *D-poset $\mathcal{D} = (D, \leq, -, 0, 1)$ is called σ-complete, if every countable subset of D has supremum and infimum.*

The equivalent structure to the D-poset is the effect algebra, which was introduce by Foulis and Bennett in the paper [2].

The important example of D-poset is MV algebra. D-poset, which is a lattice is MV algebra if and only if the following holds: $a - (a \wedge b) = (a \vee b) - b$ for each $a, b \in D$.

It is necessary define the operation product on D-poset. We use the definition, which was proposed by Kôpka in [3].

Definition 3. *Let $\mathcal{D} = (D, \leq, -, 1_D, 0_D)$ be D-poset. It is called Kôpka D-poset, if there exists the binary operation $* : D \times D \to D$, which is commutative, associative and the following statements are satisfied:*

(i) $\forall a \in D : a * 1_D = a$;

(ii) if $a, b \in D : a \leq b$, *then* $\forall c \in D$ *holds* $a * c \leq b * c$;

(iii) $\forall a, b \in D : a - (a * b) \leq 1_D - b$.

The system $\mathcal{D} = (D, \leq, -, *, 1_D, 0_D)$ is Kôpka D-poset and the operation $*$ is called the product on D-poset.

We need the definition of two important mappings the state and the observable, because we will work with the probability on D-poset.

Definition 4. *The state on D-poset $\mathcal{D}$ is a mapping $m : D \to [0, 1]$, which satisfies these rules:*

(i) $m(1_D) = 1$, $m(0_D) = 0$;

(ii) $\{a_n\}_{n=1}^{\infty} \in D$: $a_n \nearrow a$, *then* $m(a_n) \nearrow m(a)$;

(iii) $\{a_n\}_{n=1}^{\infty}$: $a_n \searrow a$, *then* $m(a_n) \searrow m(a)$.

We have to use the stronger version of state in some cases.

Definition 5. *Let $\mathcal{D}$ be a σ-complete Kôpka D-poset. The state m on D-poset $\mathcal{D}$ is called σ-aditive, if the following equality holds: $m(\bigvee_{n=1}^{\infty} a_n) = \sum_{n=1}^{\infty} m(a_n)$ for any sequence $\{a_n\}_{n=1}^{\infty}$ of elements of D such that $a_i \wedge a_j = 0$ $(i \neq j)$.*

The second important mapping in the probability theory on D-posets is the observable.

Definition 6. *Let $\mathcal{J}$ be the set of intervals $(-\infty, t); t \in R$. The observable on D-poset $\mathcal{D}$ is a mapping $x : \mathcal{J} \to D$, which satisfies:*

(i) $\{A_n\}_{n=1}^{\infty} \in \mathcal{J}$*:* $A_n \nearrow R$*, then* $x(A_n) \nearrow 1_D$*;*

(ii) $\{A_n\}_{n=1}^{\infty} \in \mathcal{J}$*:* $A_n \searrow \emptyset$*, then* $x(A_n) \searrow 0_D$*;*

(iii) $\{A_n\}_{n=1}^{\infty} \in \mathcal{J}$*:* $A_n \nearrow A$*, then* $x(A_n) \nearrow x(A)$*.*

Definition 7. *Let the mapping $x([a, b))$ is defined:*
$x([a, b)) = x((-\infty, b)) - x((-\infty, a))$.
The observable x is called the strong observable, if it holds:
$(x([a, b)) * \alpha) \wedge (x([c, d)) * \beta) = 0_D$*, for each* $\alpha, \beta \in D$ *and all intervals* $[a, b) \cap [c, d) = \emptyset$ *(*$\forall a, b, c, d \in R, \quad a \leq b, \quad c \leq d$*).*

Now we use these two mappings for the definition of the distribution function.

Definition 8. *Let the mapping $m : D \to [0, 1]$ be a state, $x : \mathcal{J} \to D$ be an observable on D-poset $\mathcal{D}$. We define the mapping $F : R \to [0, 1]$ by the following way: $F(t) = m(x((-\infty, t))), \forall t \in R$. This mapping has all properties of the distribution function.*

The important term in the probability theory is expected value.

Definition 9. *An observable $x : \mathcal{J} \to D$ is called to be integrable if there exists $E(x) = \int_R t dF(t)$; where $F : R \to [0, 1]$ is the distribution function of the observable x.*

We want to define the convergence m-almost everywhere. As first we need the definition of limes superior and limes inferior.

Definition 10. *The sequence of observables $(x_n)_{n=1}^{\infty}$ on a σ-complete D-poset $\mathcal{D}$ has limes superior (limes inferior), if there exists the mapping $\overline{x} : \mathcal{J} \to D$ ($\underline{x} : \mathcal{J} \to D$), for which holds:*

$$\overline{x}((-\infty, t)) = \bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n \left(\left(-\infty, t - \frac{1}{p} \right) \right),$$

$$\left(\underline{x}((-\infty, t)) = \bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n \left(\left(-\infty, t - \frac{1}{p} \right) \right) \right),$$

$\forall t \in R$*. It is denoted by* $\overline{x} = \limsup_{n \to \infty} x_n$ *(*$\underline{x} = \liminf_{n \to \infty} x_n$*).*

Now we can define the convergence m-almost everywhere by using limes superior and limes inferior.

Definition 11. *Let $\mathcal{D}$ be a σ-complete D-poset with a σ-aditive state m. The sequence of observables $(y_n)_{n=1}^{\infty}$ converges m-almost everywhere, if the following equality holds: $\forall t \in R$: $m(\overline{y}((-\infty, t))) = m\left(\underline{y}((-\infty, t))\right)$ and*
$F(t) = m(\overline{y}((-\infty, t)))$ is a distribution function.

Finally we define the m-preserving transformation on the set D.

Definition 12. *The mapping $\tau : D \to D$ is called m-preserving transformation, if the following statements are satisfied:*

(i) $\tau(1_D) = 1_D$*;*

(ii) $a, b, c \in D : a = b - c$ *then:* $\tau(a) = \tau(b) - \tau(c)$*;*

(iii) $a_n \in D, n \in N : a_n \nearrow a$ *then* $\tau(a_n) \nearrow \tau(a)$*;*

(iv) $m(\tau(a) * \tau(b)) = m(a * b)$*,* $\forall a, b \in D$*;*

(v) $m\left(a * \tau^i(b)\right) = m(a)\, m(b)$, $\forall a, b \in D; i = 1, 2, ...$

Theorem 1. *(Individual ergodic theorem) Let $\mathcal{D}$ be a σ-complete Kôpkov D-poset with σ-aditive state $m : D \to [0, 1]$. Let x be an integrable strong observable, the mapping τ be m-preserving transformation. Then there exists an integrable observable x^*, which satisfies the following conditions:*

(i) $E(x^*) = E(x)$;

(ii) $\frac{1}{n} \sum_{i=0}^{n-1} \tau^i \circ x \to x^*$ *m-almost everywhere.*

Acknowledgements: The paper was supported by grant VEGA 1/0539/08.

References

[1] CHOVANEC, F., KÔPKA, F.: Boolean D-posets. In: *Tatra Mountains Mathematical Publications* 10, 1993, 55-65.

[2] FOULIS, D.J., BENNETT, M.K.: In: *Effect algebras and unsharp quantum logics. Found.Phys.* 24, 1994, 1325-1346.

[3] KÔPKA, F.: D-posets with meet function. In: *Advances in Electrical and Electronic Engineering* 3, 2004, 34-36.

[4] KÔPKA, F., CHOVANEC, F.: D-lattices. In: *Inter. J. Theor. Phys.* 34, 1995, 1297-1302.

[5] KÔPKA, F., CHOVANEC, F.: D-posets. In: *Handbook of quantum logic and quantum structures*, Elsevier, 2007, 367-428.

[6] KÔPKA, F.: Quasi Product on Boolean D-posets. In: *Inter. J. Theor. Phys.* 47, 2008, 26-35.

[7] DVUREČENSKIJ, A., PULMANNOVÁ, S.: *New trends in quantum structures.* Kluwer Academic Publishers, Ister Science, Bratislava, 2000.

On counting MV algebras

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This paper is a contribution to the study of MV algebras. These algebras, introduced by Chang in [1], are studied in connection with multi-valued logic: they turn out to be the multi-valued generalization of Boolean algebras. For an introduction to MV-algebras see [3].

On counting ultrapowers of $[0,1]$

The present section is intended to continue the work of [4] on regular ultrapowers of the MV algebra $[0,1]$. For basics of model theory, including ultrafilters and ultrapowers, see [2].

The regular case

Our interest for regular ultrapowers is justified by the following theorem:

Theorem 1. *(see [4]) Let λ be an infinite cardinal. Every λ-regular ultrapower of $[0,1]$ embeds elementarily all MV chains of size at most λ.*

One can ask, for each λ, how many λ-regular ultrapowers of $[0,1]$ exist up to isomorphism. In this respect we have the following facts:

- If the continuum hypothesis is true, then up to isomorphism there is only one ultrapower of $[0,1]$ modulo a nonprincipal ultrafilter on ω (such an ultrapower is regular), see [2];
- if the continuum hypothesis is false, then there are $2^{2^{\aleph_0}}$ many ω-regular ultrapowers of $[0,1]$ up to isomorphism, see [5];
- for every ordinal α there are at least $|\alpha|$ many regular ultrapowers of $[0,1]$ over $\aleph_\alpha$ up to isomorphism, see [2] and [5].

By the previous facts, there are quite a lot of regular ultrapowers of $[0,1]$ up to isomorphism. We are interested to the structure of the class of these models. For instance, we can equip this class with the relation given by elementary embeddability. In this respect we have the following fact: if $2^\lambda = \lambda^+$ and $(\aleph_1, \aleph_0) \to (\lambda^+, \lambda)$ (for instance if $V = L$), then for every two λ-regular ultrapowers Π, Π' of $[0,1]$, Π' embeds elementarily into Π ([7]).

The general case

From [5] it follows, with an appropriate generalization, that if the Generalized Continuum Hypothesis holds, then for every uncountable cardinal κ, there are 2^{2^κ} nonisomorphic ultrapowers of $[0,1]$ modulo ultrafilters on κ (Ilijas Farah, personal communication).

On MV algebras with a given theory

On uncountable MV algebras

It is well known that every infinite Boolean algebra is unstable and has the strict order property with respect to the order (in the sense of [9]). This result extends to MV algebras:

Theorem 2. *Every infinite MV-algebra is unstable and has the strict order property, with respect to its natural (possibly partial) order.*

Corollary 1. *(Shelah) For every infinite MV algebra A and for every uncountable cardinal λ, there are 2^λ nonisomorphic MV algebras elementarily equivalent to A of cardinality λ.*

The countable case

By Corollary 1, it remains only to investigate how many *countable* MV algebras with a given first order theory exist. There is a longstanding conjecture (Vaught's conjecture, see [10]) that every complete theory, in whatever countable first order language, has either finitely many or $\aleph_0$ many or $2^{\aleph_0}$ many models. Note also that in the particular case of Boolean algebras, the number of countable models of any complete theory has been calculated in [6], and the Vaught conjecture is true in this case (every complete theory of Boolean algebras has 1 or $2^{\aleph_0}$ countable models).

For general MV algebras, quite a large part of theories has the maximum possible value. In fact, the following theorem holds.

Theorem 3. *Let A be an infinite MV algebra. Assume that for every n there is $a_n \in A$ with $na_n \neq (n+1)a_n$. Then the theory of A has $2^{\aleph_0}$ nonisomorphic countable models.*

Corollary 2. *Every infinite MV chain has a theory with $2^{\aleph_0}$ models. Every non-semisimple MV algebra has a theory with $2^{\aleph_0}$ models.*

The number of complete theories

A difference between Boolean algebras and MV algebras is the number of complete theories. In fact, in [8] it is shown that Boolean algebras have $\aleph_0$ many complete theories. In MV algebras, instead, the maximum possible value is attained (already among simple chains or chains with a fixed finite rank):

Theorem 4. *There are $2^{\aleph_0}$ complete theories of simple MV chains. The same holds for MV chains of rank n for every n.*

References

[1] C. Chang, Algebraic analysis of many-valued logics, Transactions of the American Mathematical Society 88 (1958), 467-490.

[2] C. Chang and J. Keisler, Model Theory, North-Holland, 1990.

[3] R. Cignoli, I. D'Ottaviano and D. Mundici, Algebraic foundations of many-valued reasoning, Kluiver, 2000.

[4] A. Di Nola, G. Lenzi and L. Spada, Representation of MV-algebras by regular ultrapowers of [0,1], to appear.

[5] I. Farah and S. Shelah, A dichotomy for the number of ultrapowers, Institut Mittag-Leffler, Report No. 3, 2009/2010.

[6] P. Iverson, The number of isomorphism types of complete extensions of the theory of Boolean algebras, Colloquium Mathematicum (LXII) 2 (1991), 181-187.

[7] J. Kennedy and S. Shelah, More on Regular Reduced Products, Journal of Symbolic Logic, (69) 4 (2004), 1261-1266.

[8] J. Monk, and R. Bonnet (eds), Handbook of Boolean algebras, 3 volumes, Amsterdam, North-Holland, 1989.

[9] S. Shelah, Classification theory and the number of nonisomorphic models, North-Holland, 1978.

[10] R. Vaught, Denumerable models of complete theories, Infinitistic Methods (Proc. Symp. Foundations Math., Warsaw, 1959) Warsaw/Pergamon Press (1961) pp. 303-321.

On unit intervals in L-spaces

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This is ongoing research on the class of MV-algebras which are, up to isomorphism, unit intervals in L-spaces. Recall that an L-space is a Banach lattice with an additive norm. Our aim is to prove an analogue of Kakutani's representation theorem for abstract L-spaces [4] in our MV-algebraic context and to state it as a dual categorical equivalence.

Definition 1. *By* Riesz MV-algebra *we mean an MV-algebra A toghether with an operation* $\cdot\colon [0,1]\times A \to A$ *such that*
(1) $r\cdot(x\oplus y) = (r\cdot x)\oplus(r\cdot y)$ *whenever* $x\odot y = 0$*,*
(2) $(r\oplus q)\cdot x = (r\cdot x)\oplus(q\cdot x)$ *whenever* $r\odot q = 0$*,*
(3) $r\cdot(q\cdot x) = (rq)\cdot x$*,*
(4) $1\cdot x = x$*.*

By a morphism between Riesz MV-algebras A_1 and A_2 we mean a homomorphism of MV-algebras $h\colon A_1 \to A_2$. One checks that such homomorphism preserve the $\cdot$ operation, meaning $h(r\cdot x) = r\cdot h(x)$.

In [2] it is proved that the category of Riesz MV-algebras is equivalent with the category of unital Riesz spaces (Riesz spaces with strong unit). Any Riesz MV-algebra is, up to isomorphism, the unit interval of a unital Riesz space.

We refer to [1] for all unexplained notions on Riesz spaces and L-spaces

If A is an MV-algebra and $s\colon A\to[0,1]$ is a faithful state then $\rho_s\colon A\times A\to[0,1]$ defined by $\rho_s(x,y) = s(d(x,y))$ is a metric on A. The Cauchy completion of A with respect to ρ_s is called *s-completion* and it is studied in [5], following the similar approach for ℓ-groups [3]. We denote A_s the s-completion of A.

Proposition 1. *Let A be a Riesz MV-algebra and $s\colon A\to[0,1]$ is a faithful state. Then there exists a unique L-space with strong unit (L,u) such that A_s is isomorphic with $[0,u]_L$.*

Denote $\mathbf{L}MV$ the following category:
- the objects are pairs (A,s), where A is a Riesz MV-algebra and s is a state on A such that A is s-complete,
- the morphisms are state-preserving MV-algebra homomorphisms.

Corollary 1. *The category $\mathbf{L}MV$ is equivalent with the category of unital L-spaces and norm-preserving Riesz-homomorphisms.*

Kakutani's theorem [4] states that any unital L-space can be represented as the space $L_1(\mu)$ for some measure space (X,Ω,μ).

Theorem 1 ((Kakutani)[1, 16.8])**.** *Let (L,u) be a unital L-space. Then there exists a measure space (X,Ω,μ) such that L is isometric Riesz isomorphic with $L_1(\mu)$. Moreover, (X,Ω,μ) can be chosen such that X is an extremally disconnected compact Hausdorff space, Ω is the Borel σ-algebra of X and μ is* topological *(in the sense that a Borel set D is μ-negligible if and only if it is meagre).*

For any MV-algebra A, denote $B(A)$ the boolean reduct of A.

Theorem 2. *Assume that A is a Riesz MV-algebra and $s\colon A\to[0,1]$ is a state such that A is s-complete. Let $X = Spec(B(A))$ be the spectral space of $B(A)$ and Ω the Borel σ-algebra of X. Then X is an extremally disconnected compact Hausdorff space and there is a topological probability measure $\mu\colon\Omega\to[0,1]$ such that A is isomorphic with $L_1(\mu)$. Moreover, if f is the isomorphism of A onto $L_1(\mu)$, then $\int f(a)d\mu = s(a)$ for any $a\in A$.*

We conjecture that the above result can be completed to a dual categorical equivalence.

References

[1] E. de Jonge, A.C.M. van Rooij, Introduction to Riesz spaces, Mathematical Centre Tracts 78, Amsterdam, 1977.

[2] A. Di Nola, I. Leustean, P. Flondor, MV-modules, Journal of Algebra 267 (2003), 21-40.

[3] K.R. Goodearl, D.E. Handelman, Metric completions of partially ordered abelian groups, Indiana University Mathematics Journal 29 (1980), 861-895.

[4] S. Kakutani, Concrete representations of abstract (L)-spaces and the mean ergodic theorem, Annals of Mathematics 42 (2) (1941), 523-537.

[5] I. Leustean, Metric Completions of MV-algebras with states: An approach to stochastic independence, Journal of Logic and Computation, doi:10.1093/logcom/exp016.

From Linguistic Models of Vagueness to t-Norm Based Fuzzy Logic

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Adequate models of reasoning with vague information are not only of perennial interest to philosophers and logicians (see, e.g., [7, 12, 11] and references there), but also in the focus of linguistic research (see, e.g., [10, 8, 1]). Of particular interest from a logical point of view are approaches to formal semantics of a natural language that can be traced back to Richard Montague's ground braking work, firmly connecting modern formal logic and linguistics (see, e.g., the handbook chapter [9] and the widely used textbook [6]). At a first glimpse, it seems that all important contemporary linguistic models of vagueness are *incompatible* with the degree based approach offered by fuzzy logic (see, e.g., [5]). E.g., Manfred Pinkal in his frequently cited (and translated) monograph [10] explicitly argues that many-valued, truth functional logics are inadequate for modelling central linguistic phenomena of vagueness and indeterminateness. One of the key points here is *truth functionality*: consider e.g. the following two sentences:

$$\textit{The sky is blue or the sky is not blue.} \tag{3}$$

$$\textit{The sky is blue or the sky is blue.} \tag{4}$$

If we assign the truth value 0.5 to both $blue(sky)$ and $\neg blue(sky)$, then these sentences (1) and (2) will receive the same truth value in any (truth functional) fuzzy logic. This goes, as Pinkal argues, completely against human intuition.

More specifically, contemporary linguists seem to agree that a special type of *context dependency* is the key to understand the semantics of vague predicates ('tall', 'nice', 'is a heap', 'enjoys', 'likes', ...), but also of corresponding predicate modifiers ('very', 'definitely', ...) and quantifiers ('most', 'many', 'few', ...). However, a closer look at corresponding recent papers on vagueness, in particular [1, 8], reveals that contexts are primarily used to keep track of varying *standards of assertability* connected with *gradable predicates*. This observation is our starting point in exploring connections between fuzzy logic and the cited linguistic models of vagueness.

We will show how *fuzzy sets* and *fuzzy relations* can be systematically extracted from a given context space endowed with a probability measure (or more generally, possibility measure) intended to model the relative salience and plausibility of different contexts (standards). Roughly speaking, the membership degree of an individual **a** (say 'Adam') in a fuzzy set modelling a predicate **T** (say 'is tall') gets identified with the probability — alternatively: degree of possibility or degree of necessity — that **a** satisfies the assertability standard associated with **T** in a randomly chosen context. In this manner t-norm and co-t-norm based operators re-emerge as semantic correlates of conjunction, disjunction, and other logical connectives, if one insists on global evaluations that ignore all dependencies between context specific standards pertaining to different predicates. In contrast, local evaluations, i.e. those referring to individual contexts, lead to an *intensional semantic framework*, also for logical connectives. While an intensional evaluation, based on a specific context space, allows to model phenomena of vague language [1, 8] that escape the coarser truth functional approach of fuzzy logic, the price to be paid for the more fine grained analysis is higher computational complexity. In this respect, t-norm based truth functions can be seen as *efficient extensional approximations* to potentially very complex *intensional evaluations* with respect to context dependent assertability conditions. We already presented these approximations at the Workshop on Uncertainty Processing earlier last year [2].

On the other hand, one can strive for other justifications for using concrete t-norm based fuzzy logics based on the notions of linguistic contexts. For example, Giles's Game [4] is a combined dialogue/betting game originally motivated for reasoning in physical theories and based on binary experiments. We will see how to re-interpret the game in the presence of vague adjectives and Barker's linguistic contexts. This way Barker's definitions for predicate modifiers such as *very, definitely, clearly* can be illustrated in terms of the game.

Another link to fuzzy logic gets apparent when focusing on the *similarity* between different contexts. We start with the observation that the individual worlds that form a concrete context are to a higher or lesser degree similar to each other. After all, vagueness, in this model, amounts to the fact that while hearers don't have access to precise criteria for judging a statement as definitely true or false, they are nevertheless supposed to evaluate with respect to a given set of such precise criteria that is constrained in a specific manner reflecting the context of discourse. Taking the degrees of similarity between the worlds as a basis of an evaluation that is graded accordingly, provides a further connection between contextual models and fuzzy logic.

References

[1] Ch. Barker: The Dynamics of Vagueness, *Linguistics and Philosophy* 25(1), 2002, 1–36.

[2] C.G. Fermüller, C. Roschger: Bridges between contextual linguistic models of vagueness and t-norm based fuzzy logic. *Proceedings of the eight Workshop on Uncertainty Processing*, T.Kroupa, J.Vejnarova eds.,69-79,2009

[3] C.G. Fermüller, C. Roschger: Bridges between contextual linguistic models of vagueness and fuzzy logic. *International Journal of Approximate Reasoning*, submitted.

[4] R. Giles: A non-classical logic for physics. *Studia Logica* 33(4):397-415, 1974.

[5] P. Hájek: *Metamathematics of Fuzzy Logic*. Kluwer, 1998.

[6] I. Heim, A. Krätzer: *Semantics in Generative Grammar*. Blackwell, 2008 (1998).

[7] R. Keefe, P. Smith (eds.): *Vagueness – A Reader*, MIT Press, 1987.

[8] Ch. Kennedy: Vagueness and Grammar: The semantics of Relative and Absolute Gradable Adjectives. *Linguistics and Philosophy* 30(1), 2007, 1–45.

[9] B. Partee, H.L.W. Hendriks: Montague Grammar. In J. van Benthem, A. ter Meulen (eds), *Handbook of logic and Language*, Elsevier Science and The MIT Press, 1996, 5–91.

[10] M. Pinkal: *Logic and Lexicon*, Studies in Linguistics and Philosophy, Vol. 56, Kluwer Academic Publishers, 1995.

[11] St. Shapiro: *Vagueness in Context*, Oxford University Press, 2006.

[12] T. Williamson: *Vagueness*, Routledge, 1996.

Semiring and semimodule issues in MV-algebras

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Based on [6], joint work with Antonio Di Nola.

We present a new perspective on the theory of MV-algebras by drawing a line connecting such structures — and the categorically equivalent ones of lattice-ordered Abelian groups with strong unit — with the theory of idempotent semirings and its ramifications.

The theory of idempotent semirings is nowadays well-established and boasts a wide range of applications in many fields, such as discrete mathematics, computer science, computer languages, linguistic problems, finite automata, optimization problems, discrete event systems, computational problems et cetera. The theory arising from the substitution of the fields of real and complex numbers with idempotent semirings and/or semifields is often referred to as *idempotent* or *tropical mathematics*.

As Litvinov observed in [7], "idempotent mathematics can be treated as the result of a dequantization of the traditional mathematics over numerical fields as the Planck constant $\hbar$ tends to zero taking imaginary values." This point of view was also presented, by Litvinov himself and Maslov, in [8]. Another equivalent presentation of idempotent mathematics is as an asymptotic version of the traditional mathematics over the fields of real and complex numbers. This idea is expressed in terms of an idempotent correspondence principle which is closely related to the well-known correspondence principle of N. Bohr in quantum theory. In fact, many important and useful constructions and results of the traditional mathematics over fields correspond to analogous constructions and results over idempotent semirings and semifields; to this extent the aforementioned paper [7] provides an impressive list of references.

Another important aspect of the development of such a theory is the linear algebra and the algebraic geometry of idempotent semirings, better known as *tropical geometry*, whose most important model is the geometry of the tropical semifield $\langle \overline{\mathbb{R}}, \min, +, \infty, 0 \rangle$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$: its objects are polyhedral cell complexes which behave like complex algebraic varieties.

A connection between MV-algebras and a special category of additively idempotent semirings was first observed in [3] and, eventually, enforced in [2]. On the one hand, every MV-algebra has two *semiring reducts* isomorphic to each other by the involution *; on the other hand, the category of *MV-semirings*, defined in [2], is isomorphic to the one of MV-algebras. Such results led to interesting applications of MV-semirings and their semimodules to the theory of fuzzy weighted automata [14], and to an algebraic approach to fuzzy compression algorithms [4, 5] and mathematical morphological operators [13] for digital images.

Another link between MV-algebras and semiring theory relies on the well-known and celebrated categorical equivalence of such algebras with lattice-ordered Abelian groups with strong unit [9] or — that is the same, as we are going to see in details — with idempotent semifields with strong unit.

The main results we present can be briefly summarized as follows.

- A representation for homomorphisms of free semimodules — which, for finitely generated free semimodules, is analogous to the matrix representation of linear maps in classical linear algebra — and its generalization to all semimodule homomorphisms.
- A matrix-based characterization of finitely generated projective semimodules over any semiring.
- A characterization of cyclic projective MV-semimodules as direct summands of the free cyclic one.
- We introduce tropical polynomials over MV-algebras; moreover we describe such polynomials in the case of the standard MV-algebra $[0, 1]$ and their connection with McNaughton functions.
- A functional representation of any MV-algebra as a subsemiring of a semiring of endomorphisms. What is outstanding, here, is the fact that the MV-algebraic multiplication and sum, both commutative, are represented as endomorphism composition — an operation which is typically non-commutative.

- The relationship between MV-semimodules and semimodules over idempotent semifields with strong unit as a consequence both of Mundici categorical equivalence and of constructions and results, of more general interest for idempotent semirings, that we shall present. In particular we shall see that MV-semimodules can be obtained by means of a sort of truncation of semimodules over idempotent semifields, as well as MV-algebras are obtained by truncating idempotent semifields.

- The construction of the Grothendieck groups of semirings and MV-algebras following the classical ring-theoretic construction; such a construction is easily proven, also thanks to the aforementioned characterization of finitely projective semimodules, to be functorial.

All these results broaden the already wide variety of connections between MV-algebras and other theories. Our intention is mainly to establish such new links so as to open new research lines and motivations for future works on this matter. Indeed, such a semiring-theoretic perspective on MV-algebras naturally suggests many questions and ideas.

For example, one may ask if it is possible to define tropical algebraic varieties on MV-algebras as a "truncated" version of the ones defined on the tropical semifield of the reals. Moreover, if such a question has a positive answer, it would be reasonable to ask whether there would be any connection between such a theory and the well-established geometric theory of MV-algebras (see, for instance, [1, 10, 11, 12]).

Another issue that naturally arises is related to the functor K_0 associating an Abelian group to every MV-algebra. Obviously, such a functor immediately suggests the development of an algebraic K-theory of MV-algebras which, however, needs to be strongly motivated, i. e. is expected to advance the knowledge of MV-algebras rather than being a pure speculative exercise.

Besides all these possible advances, it is unquestionable that the strong tie between semiring and semimodule theories of MV-algebras and idempotent semifields — whose common DNA lies on Mundici categorical equivalence — is worth to be investigated as deeply as possible. As a matter of fact, such an equivalence directly relates the tropical semifield $\langle\overline{\mathbb{R}}, \min, +, \infty, 0\rangle$, which is the basis for the most important concrete model of tropical geometry, with the MV-algebra $[0, 1]$, that generates the variety of MV-algebras.

References

[1] Aguzzoli S., *Geometric and Proof-Theoretic Issues in Łukasiewicz Propositional Logics*, Ph.D. dissertation, University of Siena, 2002.

[2] Belluce L. P., Di Nola A., Commutative Rings whose Ideals form an MV-algebra, *Mathematical Logic Quarterly*, **55** (5), 468-486, 2009.

[3] Di Nola A., Gerla B., Algebras of Łukasiewicz's Logic and their Semiring Reducts, *Idempotent Mathematics and Mathematical Physics*, 131-144, Contemp. Math., **377**, Amer. Math. Soc., 2005.

[4] Di Nola A., Russo C., Łukasiewicz Transform and its application to compression and reconstruction of digital images, *Informat. Sci.*, **177** (6), 1481-1498, 2007.

[5] Di Nola A., Russo C., Łukasiewicz Transform Based Algorithm for Image Processing, *Proc. of the 2006 IEEE Intl. Conf. on Fuzzy Systems*, pp. 1996-2003, Vancouver, BC, Canada, July 16–21, 2006.

[6] Di Nola A., Russo C., Semiring and semimodule issues in MV-algebras, submitted.
`arXiv:1002.4013v1 [math.RA]`

[7] Litvinov G. L., Maslov dequantization, idempotent and tropical mathematics: a brief introduction, *Journal of Mathematical Sciences*, **140** (3), 2007.

[8] Litvinov G. L., Maslov V. P., Correspondence principle for idempotent calculus and some computer applications, Preprint IHES/M/95/33, Institut des Hautes Études Scientifiques, Bures-sur-Yvette 1995.
`arXiv:0101021v1 [math.GM]`

[9] Mundici D., Interpretation of *AF* C^*-algebras in Łukasiewicz sentential calculus, *J. Functional Analysis*, **65**, 15-63, 1986.

[10] Mundici D., A Constructive Proof of McNaughton's Theorem in Infinite-Valued Logic, *Journal of Symbolic Logic*, **59**, 596-602, 1994.

[11] Mundici D., Łukasiewicz normal forms and toric desingularizations, In: W. Hodges et al. (eds.), *Proceedings of Logic Colloquium 1993*, Keele, 401-423, Oxford University Press, 1996.

[12] Panti G., A geometric proof of the completeness of the Łukasiewicz calculus, *Journal of Symbolic Logic*, **60**, 563-578, 1995.

[13] Russo C., A Unified Algebraic Framework for Fuzzy Image Compression and Mathematical Morphology, In: V. Di Gesù, S.K. Pal, and A. Petrosino (eds.), Proc. of WILF 2009, *Lecture Notes in Artificial Intelligence*, **5571**, 205-212, Springer-Verlag, 2009.
`arXiv:1002.0982v1 [cs.IT]`

[14] Schwarz S., Lukasiewicz Logic and Weighted Logics over MV-Semirings, *J. Autom. Lang. Comb.*, **12** (4), 485-499, 2007.

Omitting type theorems for Łukasiewicz logic

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The Omitting Type Theorem is an important tool in model theory, it belongs to the general class of tools for constructing models. The theorem was first found in the contest of arithmetic and ω-consistency in [6, 9]. The result was further developed by C.C.Chang in [1]. I will give two omitting type theorems for Łukasiewicz logic, following the lines of [7] and [8].

Preliminaries

Due to space constraints I will not give any standard definition; the reader is advised to consult [2] for all unexplained notions about Łukasiewicz propositional logic and its algebraic counterpart: MV-algebras; for Łukasiewicz predicate logic the reader may consult [5]. Finally, an up-to-date exposition on predicate fuzzy logics, can be found in [3]. All structures considered henceforth are *standard*.

In Łukasiewicz predicate logic it is possible to define a total hierarchy on the lines of the *arithmetical hierarchy* in classical logic. As in the classical case, one has $\Sigma_n \cup \Pi_n \subseteq \Sigma_{n+1} \cap \Pi_{n+1}$. Given a theory T, the set of its Σ_n consequences (Π_n consequences) is indicated by $T_{\exists_n}$ ($T_{\forall_n}$, respectively).

Let $\mathcal{M}$ be an A-structure, $\mathcal{L}(\mathcal{M})$ is the expansion of the language $\mathcal{L}$ with a new constant symbol for each element of M. The **diagram** of $\mathcal{M}$, i.e. the set of atomic formulae φ in $\mathcal{L}(\mathcal{M})$ such that $\|\varphi\|_{\mathcal{M}} = 1$ (equivalently written as $\mathcal{M} \models \varphi$), is indicated by $D(\mathcal{M})$; $\mathrm{Th}(\mathcal{M})$ is the set of formulae φ such that $\|\varphi\|_{\mathcal{M}} = 1$.

In order to obtain a result similar to the one by Chang, in the contest of Łukasiewicz logic, a slightly sharper version of the forcing introduced in [4] is needed.

Definition 1. *Let $\mathcal{M}_1 \subseteq \mathcal{M}_2$ be A-structures, then $\mathcal{M}_1$ is an n-**substructure** of $\mathcal{M}_2$, in symbols $\mathcal{M}_1 \preccurlyeq_n \mathcal{M}_2$, if for any Π_n sentence φ, it holds $\mathcal{M}_1 \models \varphi$ iff $\mathcal{M}_2 \models \varphi$.*

*If $n = 0$ then $\mathcal{M}_1$ is called **substructure** of $\mathcal{M}_2$. If $\mathcal{M} \preccurlyeq_n \mathcal{M}_2$ is true for any n then $\mathcal{M}_1$ is an **elementary substructure of** $\mathcal{M}_2$, written $\mathcal{M}_1 \preccurlyeq \mathcal{M}_2$*

Let T a theory of $\mathcal{L}$, $\mathrm{Mod}_n(T)$ indicates the class of models of $T_{\forall_{n+1}}$. With the above notation one has $\mathcal{M} \in \mathrm{Mod}_n(T)$ iff there exists $\mathcal{M}'$ such that $\mathcal{M}' \models T$ and $\mathcal{M} \preccurlyeq_{n+1} \mathcal{M}'$

Without going into the details (see [4]) the new version of forcing can be obtained by substituting the notion of substructure with the stronger n-substructure. This leads to a refined notion of forcing called the **n-forcing value of φ at $\mathcal{M}$**, in symbols $[\varphi]^n_{\mathcal{M}}$. Such a new definition is justified by an easy generalisation of the following result.

Theorem 1 ([10]). *A theory is inductive (i.e. it is closed under limits of chains) if, and only if, it is equivalent to a Π_2 theory.*

Indeed it can be proved that T is Π_{n+2} axiomatisable if, and only if, it is closed under unions of n-chains.

Omitting type theorems

In the following the language is assumed to be countable.

Definition 2. *Let $\mathcal{M} \in \mathrm{Mod}_n(T)$ then $\mathcal{M}$ is $\mathrm{Mod}_n(T)$-**generic** if for any sentence φ of $L(\mathcal{M})$, $[\varphi]^n_{\mathcal{M}} \oplus [\neg\varphi]^n_{\mathcal{M}}$=1.*

The proof of the subsequent lemmas can be easily extracted from the proofs in [4].

Lemma 1. *Let $\mathcal{M} \in \mathrm{Mod}_n(T)$ then it exists $\mathcal{M}^* \in \mathrm{Mod}_n(T)$ such that $\mathcal{M} \preccurlyeq_n \mathcal{M}^*$ and $\mathcal{M}^*$ is $\mathrm{Mod}_n(T)$-generic.*

Lemma 2. *Let $\mathcal{M}, \mathcal{M}' \in \mathrm{Mod}_n(T)$, then*

1. $\mathcal{M}$ is $\mathrm{Mod}_n(T)$-generic iff for any sentence φ one has $[\varphi]^n_{\mathcal{M}} = \|\varphi\|_{\mathcal{M}}$.

2. If $\mathcal{M}, \mathcal{M}'$ are $\mathrm{Mod}(T, m)$-generic, and $\mathcal{M} \preccurlyeq_n \mathcal{M}'$ then $\mathcal{M} \preccurlyeq \mathcal{M}'$.

3. If $\mathcal{M}$ is $\mathrm{Mod}_n(T)$-generic, $\mathcal{M} \preccurlyeq_n \mathcal{M}'$ and φ is a Π_{n+2}-sentence of $L(\mathcal{M})$ then $\|\varphi\|_{\mathcal{M}'} \leq \|\varphi\|_{\mathcal{M}}$

4. $T_{\forall_{n+2}} \subseteq T^{(F,n)}$ where $T^{(F,n)}$ is the set of all sentences of $\mathcal{L}$ valid in all the $\mathrm{Mod}_n(T)$-generic models.

Theorem 2 (Main Lemma)**.** *Let $\mathcal{M}$ be $\mathrm{Mod}_n(T)$-generic and φ be a Σ_{n+2}-sentence of $L(\mathcal{M})$, suppose that $\mathcal{M} \models \varphi$ then there exists a Σ_{n+1}-sentence ψ of $L(\mathcal{M})$ such that:*

1. $\mathcal{M} \models \psi$,

3. $T \models \psi \to \varphi$,

3. all the constants of $\mathcal{M}$ which occur in ψ already occur in φ.

Definition 3. *A set Γ of formulae of $\mathcal{L}$ is called a **type** if all the formulae in Γ are consistent with T and it is closed under $\wedge$. A type Γ is called a Σ_n**-type** if all the formulae in Γ are equivalent to a Σ_n-formula.*

*If Δ, Γ are types, I write $\Delta \leq \Gamma$ if $T \cup \Delta \models \gamma$ for all $\gamma \in \Gamma$. A Σ_{n+1}-type is called a $(n+1)$**-existential type** if there exist no Σ_n-type Δ such that $\Delta \leq \Gamma$.*

The following is an easy corollary of 2.

Theorem 3. *If $\mathcal{M}^*$ is a $\mathrm{Mod}_n(T)$-generic model and $\mathcal{M}^* \models T$, then $\mathcal{M}^*$ omits all the $(n+2)$-existential types.*

Theorem 4 (First Chang omitting type theorem)**.** *Let T be a theory of $\mathcal{L}$, such that $T \subseteq \Pi_{n+2}$. For any model $\mathcal{M}$ of T, there exists an extension $\mathcal{M}^*$ of $\mathcal{M}$ such that:*

1. $\mathcal{M}^*$ is a model of T,

2. $\mathcal{M}^*$ realises every Σ_{n+1}-type,

3. $\mathcal{M}^*$ omits every type $(n+2)$-existential.

Definition 4. *Let Γ a set of formulae of $\mathcal{L}$, let $\mathbf{x}$ be the seqeence of all the variables which appear in Γ. We will say that Γ is a ∞**-universal type** if it satisfies the following conditions:*

1. All the formulae in Γ are consistent with T,

2. there exists an enumeration $(\gamma_k)_{k \in \omega}$ of Γ such that $T \models \gamma_{k+1} \to \gamma_k$ for any $k \in \omega$,

3. There exists a strictly increasing function f such that $\gamma_k \in \Pi_{f(k)} - \Sigma_{f(k)}$ for any $k \in \omega$,

4. There exists no type Δ and an integer k_0 such that: for any $k \geq k_0$ there exists a subset $\Delta_k \subseteq \Delta$ such that Δ_k is T-equivalent to a set of formulae in $\Sigma_{f(k)}$ and $T \cup \Delta_k \models \gamma_k$.

Theorem 5 (Second Chang omitting type theorem)**.** *Let T be a theory of $\mathcal{L}$. For any model $\mathcal{M}$ of T, there exists an extension $\mathcal{M}^*$ of $\mathcal{M}$ such that:*

1. $\mathcal{M}^*$ is a model of T,

2. $\mathcal{M}^*$ omits every ∞-universal type.

References

[1] C. C. Chang. Omitting types of prenex formulas. *Journal of Symbolic Logic*, 32:61–74, 1967.

[2] R. Cignoli, I.M.L D'Ottaviano, and D. Mundici. *Algebraic Foundations of Many-valued Reasoning*, volume 7 of *Trends in Logic, Studia Logica Library.* Kluwer Academic, 2000.

[3] P. Cintula and P. Hájek. Triangular norm based predicate fuzzy logics. *Fuzzy Sets and Systems*, 2009.

[4] A. Di Nola, G. Georgescu, and L. Spada. Forcing in Łukasiewicz predicate logic. *Studia Logica*, 89(1):111–145, May 2008.

[5] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic, Studia Logica Library.* Kluwer Academic, Berlin, 1998.

[6] L. Henkin. A Generalization of the Concept of ω-Consistency. *The Journal of Symbolic Logic*, 19(3):183–196, 1954.

[7] H. J. Keisler. Forcing and the omitting types theorem. In M. Morley, editor, *Studies in Model Theory*, MAA Studies in Math, pages 96–133. The Mathematical Association of America, Buffalo, N.Y., 1973.

[8] J.-C. Lablaniquie. Forcing infini et theoreme d'omission des types de Chang. *Ann. Sci. Univ. Clermont, Ser. Math.*, 16:27–31, 1978.

[9] S. Orey. On ω-consistency and related properties. *The Journal of Symbolic Logic*, 21:246–252, 1956.

[10] L. Spada. Some consequences of compactness in Łukasiewicz logic. In C. Dross, P. Peppas, and C. Tsinakis, editors, *Proceedings of the 7th Panhellenic Symposium of Logic*, pages 162–167. Patras University press, 2008.

Unitary Unification in Subvarieties of WNM-Algebras

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A *commutative integral bounded residuated lattice* is an algebra $A = (A, \wedge, \vee, \odot, \rightarrow, \bot, \top)$ of type $(2, 2, 2, 2, 0, 0)$ such that $(A, \wedge, \vee, \bot, \top)$ is a bounded lattice, with top $\top$ and bottom $\bot$, $(A, \odot, \top)$ is a commutative monoid, and the *residuation* equivalence, $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$, holds. Commutative integral bounded residuated lattice form an algebraic variety [6]. If the lattice order is total, A is called a *chain*. An *MTL-algebra* is a commutative integral bounded residuated lattice satisfying the *prelinearity* equation, $(x \rightarrow y) \vee (y \rightarrow x) = \top$. The unary term operation $\neg x$ is defined by $x \rightarrow \bot$, and the binary term operation $x \leftrightarrow y$ is defined by $(x \rightarrow y) \odot (y \rightarrow x)$. A *WNM-algebra* is an MTL-algebra satisfying the *weak nilpotent minimum* equation,

$$\neg(x \odot y) \vee ((x \wedge y) \rightarrow (x \odot y)) = \top.$$

In this note, we investigate the *unification type* of certain subvarieties of WNM-algebras, namely *NMG-algebras*, *NM-algebras*, and *RDP-algebras*, respectively characterized by the additional equations,

$$\neg\neg x \rightarrow x = \top,$$
$$(\neg\neg x \rightarrow x) \vee ((x \wedge y) \rightarrow (x \odot y) = \top,$$
$$\neg\neg x \vee (x \rightarrow \neg x) = \top.$$

In every WNM-algebra, the operation $\vee$ and the constant $\top$ are definable as term operations over $\wedge$, $\odot$, $\rightarrow$, $\bot$ [10].

It is known that the aforementioned varieties are singly generated [5, 13, 12]. In particular, the variety of NM-algebras is singly generated by the algebra $[0, 1]_{NM} = ([0, 1], \wedge^{NM}, \odot^{NM}, \rightarrow^{NM}, \bot^{NM})$, defined by $\bot^{NM} = 0$, $\wedge^{NM}(x, y) = \min\{x, y\}$, and, for every $x, y \in [0, 1]$,

$$x \odot^{NM} y = \begin{cases} 0 & \text{if } x \leq \neg y, \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$
$$x \rightarrow^{NM} y = \begin{cases} 1 & \text{if } x \leq y, \\ \max\{\neg x, y\} & \text{otherwise.} \end{cases}$$

The variety of NMG-algebras is singly generated by the algebra $[0, 1]_{NMG} = ([0, 1], \wedge^{NMG}, \odot^{NMG}, \rightarrow^{NMG}, \bot^{NMG})$, defined by $\bot^{NMG} = 0$, $\wedge^{NMG}(x, y) = \min\{x, y\}$, and, for every $x, y \in [0, 1]$ and some (arbitrary) fixed $0 < a < 1$:

$$x \odot^{NMG} y = \begin{cases} \min\{x, y\} & \text{if } a < x + y, \\ 0 & \text{otherwise.} \end{cases}$$
$$x \rightarrow^{NMG} y = \begin{cases} 1 & \text{if } x \leq y, \\ \max\{a - x, y\} & \text{otherwise.} \end{cases}$$

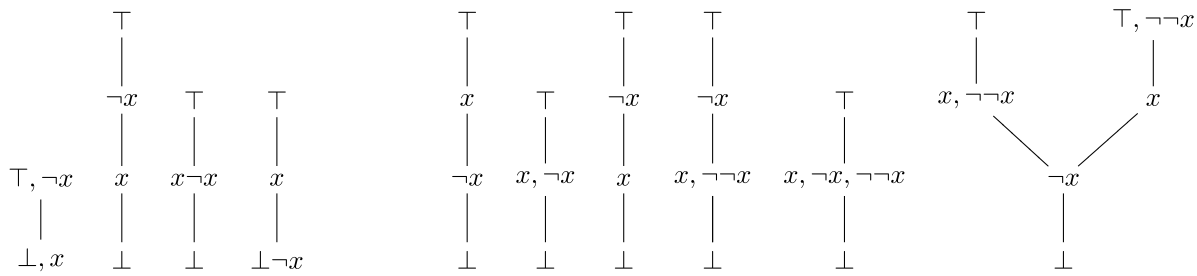
The variety of RDP-algebras is singly generated by the algebra $[0, 1]_{RDP} = ([0, 1], \wedge^{RDP}, \odot^{RDP}, \rightarrow^{RDP}, \bot^{RDP})$, defined by $\bot^{RDP} = 0$, $\wedge^{RDP}(x, y) = \min\{x, y\}$, and, for every $x, y \in [0, 1]$ and some (arbitrary) fixed $0 < a < 1$:

$$x \odot^{RDP} y = \begin{cases} 0 & \text{if } x, y \leq a, \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$
$$x \rightarrow^{RDP} y = \begin{cases} 1 & \text{if } x \leq y, \\ a & \text{if } y < x \leq a, \\ y & \text{otherwise.} \end{cases}$$

In the sequel, $L \in \{NM, NMG, RDP\}$. Notice that the operation $\to_L$ is the unique binary operation over the real interval $[0,1]$ satisfying the residuation equivalence with respect to $\odot_L$. This observation suggests a logical interpretation of the operations $\odot^L$, $\to^L$, $\neg^L$, as the conjunction, implication, and negation of a propositional many-valued *L-logic*, having $[0,1]_L$ as its complete semantics: an L-term t is a *tautology* of L-logic, if and only if the equality $t^L = \top^L$ holds in $[0,1]_L$. In equivalent, syntactic terms, the L-logic lies in the established hierarchy of schematic extensions of MTL-logic, the logic of all left-continuous triangular norms and their residuals [8].

By universal algebraic facts [3], the free n-generated L-algebra, $F_n(L)$, is the clone of n-ary term operations of the algebra $[0,1]_L$, equipped with operations defined pointwise by the fundamental operations of $[0,1]$. [1] In the logical setting, the algebra $F_n(L)$ is the Lindenbaum-Tarski algebra of the n-variate fragment of the L-logic.

Notice that $F_n(L)$ is finite, because the variety of L-algebras is locally finite. Indeed, the subdirectly irreducible members of subvarieties of MTL-algebras are chains [5], and WNM-chains are locally finite, thus the variety of WNM-algebras is locally finite [10]. It follows that the variety of L-algebras is locally finite, thus allowing for a combinatorial representation of free finitely generated L-algebras, as suitable algebras of maximal antichains over certain posets [1, 11]. See Figure 1 for the case $n = 1$.



(a) $F_1(RDP)$. (b) $F_1(NM)$. (c) $F_1(NMG)$.

Figure 1: For $L \in \{NM, NMG, RDP\}$, $F_1(L)$ is a suitable algebra of maximal antichains over the depicted forest.

In this note, we settle the unification type for the L-unification problem, defined as follows. Let $T_n(L)$ denote the L-algebra of terms over the variables $x_1, \ldots, x_n$. An instance to the L-unification problem is a term $t \in T_n(L)$, and the question is whether there exists a *unifier* for t, that is, an endomorphism h of $T_n(L)$ such that the equality

$$h(t)^L = \top^L$$

holds in L-algebras; note that h is uniquely determined by its restriction to $x_1, \ldots, x_n$. A unifier h for $t \in T_n(L)$ such that $h(x_i) \in \{\bot, \top\}$ for $i = 1, \ldots, n$ is said *ground*. It is easy to check that a term $t \in T_n(L)$ is unifiable if and only if it has a ground unifier.

Let h and h' be unifiers for t. Then, h' is *less general* than h, in symbols $h' \leq h$, if there exists an endomorphism h'' of $T_n(L)$ such that the equality

$$(h'(x_i) \leftrightarrow h''(h(x_i))^L = \top^L$$

for $i = 1, \ldots, n$ holds in L-algebras. A unifier h for t such that every unifier for t is less general than h is said a *most general* unifier for t. The *type* of L-unification is *unitary*, if every unifiable L-term has a most general unifier (this is the nicest possible type in unification theory, since in general unifiable terms may have finitely or infinitely many incomparable unifiers).

Relying upon the combinatorial representation of free finitely generated RDP-algebras, the authors recently proved (constructively) that the RDP-unification is unitary [2]. A straightforward, if tedious, computation allows to extend this result to NM and NMG, further generalizing previous work of Dzik on subvarieties of BL-algebras [4].

[1] The clone of n-ary term operations over $[0,1]$ is the smallest set of n-ary operations over $[0,1]$ containing the n-ary projections $x_1, \ldots, x_n$, and closed under arbitrary compositions with the fundamental operations of the generic algebra.

Proposition 1. *Let $L \in \{NM, NMG, RDP\}$, let $t \in T_n(L)$, and let g be a ground unifier for t. Then, the endomorphism h of $T_n(L)$ defined by,*

$$h(x_i) = ((t^2 \to x_i) \odot (\neg t^2 \to g(x_i))),$$

for $i = 1, \ldots, n$, is the most general unifier for t.

Corollary 1. *Let $L \in \{NM, NMG, RDP\}$. The L-unification type is unitary.*

In view of the connection between unification and admissibility identified by Ghilardi [7], and the recent work of Dzik [4] and Jeřábek [9] on substructural and many-valued logics along this line, an interesting and natural development of the present note is the characterization of admissible rules in the aforementioned subvarieties of WNM-algebras.

References

[1] S. Aguzzoli and B. Gerla. Normal Forms and Free Algebras for Some Extensions of MTL. *Fuzzy Sets and Systems*, 159(10):1131–1152, 2008.

[2] S. Bova and D. Valota. Finitely Generated RDP-Algebras: Spectral Duality, Finite Coproducts and Logical Properties. *Submitted.*

[3] S. Burris and H.P. Sankappanvar. *A Course in Universal Algebra.* Springer-Verlag, 1981.

[4] W. Dzik. Unification in Some Substructural Logics of BL-Algebras and Hoops. *Reports on Mathematical Logic*, 43:73–83, 2008.

[5] F. Esteva and L. Godo. Monoidal t-Norm Based Logic: Towards a Logic for Left-Continuous t-Norms. *Fuzzy Sets and Systems*, 124(3):271–288, 2001.

[6] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics.* Elsevier, 2007.

[7] S. Ghilardi. Unification in Intuitionistic Logic. *J. Symb. Log.*, 64(2):859–880, 1999.

[8] S. Jenei and F. Montagna. A Proof of Standard Completeness for Esteva and Godo's Logic MTL. *Studia Logica*, 70(2):183–192, 2002.

[9] E. Jeřábek. Admissible rules of Łukasiewicz logic. *Journal of Logic and Computation*, 20(2):425–447, 2010.

[10] C. Noguera, F. Esteva, and J. Gispert. On Triangular Norm Based Axiomatic Extensions of the Weak Nilpotent Minimum logic. *Mathematical Logic Quarterly*, 54(4):387–409, 2008.

[11] D. Valota. Poset Representation for Free RDP-Algebras. In H. Hosni and F. Montagna, editors, *Probability, Uncertainty and Rationality*, volume 10 of *CRM Series.* Edizioni della Scuola Normale Superiore, Pisa 2010.

[12] S. Wang. A fuzzy logic for the revised drastic product t-norm. *Soft computing*, 11(6):585–590, 2007.

[13] S. Wang, B. Wang, and D. Pei. A fuzzy logic for an ordinal sum t-norm. *Fuzzy Sets and Systems*, 149:297–307, 2005.