Proof Search in Hájek's Basic Logic BL

Simone Bova
University of Siena, Italy
bova@unisi.it

Coauthors: Franco Montagna (Università degli Studi di Siena, Italy)

We study the logical and computational properties of a proof system for Hájek's Basic Logic BL, called RWBL. The system enjoys nice properties both in the proof-theoretical and in the proof-complexity perspective, allowing for automated search of BL tautology proofs.

From the proof-theoretical point of view, we prove that the rules of the underlying logical calculus, called RHBL, are sound and invertible, and hence the system is sound and complete. Moreover, RHBL is analytic since it is cut-free and it owns the subformula property, and hence the system is suitable for automated proof searching. It is worth noting that the logical calculus RHBL is based on a relational hypersequents framework, which was first applied by Ciabattoni, Fermüller and Metcalfe (2004) to the fundamental schematic extensions of BL.

From the proof-complexity point of view, we accomplished to lower the proof tree size as compared to, as far as we know, all the other proof systems published for BL, namely the tableaux calculus presented by Montagna, Pinna and Tiezzi (2003) and the sequent calculus presented by Aguzzoli (2004). In fact, the former calculus has the benefit of retaining control over the width of the proof tree, but not over the height, whereas the latter calculus retains control over the height of the proof tree, but not over the width.

The present system combines both these features. Specifically, if A is a formula with n connectives, then A has a RWBL proof tree with height $\leq n$ and width $\leq 5^n$. This fact lowers immediately the upper bounds of the provability problem of BL, as can be derived from the previously mentioned calculi. Specifically, we obtain that the size complexity of a BL tautology proof, as well as the time complexity of the algorithm for searching such a proof (dominated by the proof size), are upper bounded by $O(5^n)$. Moreover, if a formula is unprovable, we can exploit the constructiveness of the polynomial time algorithm for leaves validity, in order to provide a procedure to build countermodels in the underlying semantics.