Evaluation games in Łukasiewicz logic

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The idea of a connection between game theory and logic can be traced back to the 1950s (Paul Lorenzen's dialogue games). Nevertheless the current boom in game semantics began in the 1980s with the work of Jaakko Hintikka and his followers. They introduced and developed the system of independent friendly (IF) logics, the semantics of which extends evaluation games for classical logic using the concept of a game of imperfect information. Since then game semantics has been extensively studied and a number of logics (e.g. intuitionistic, modal, linear) have received a game theoretical interpretation.

The using of game theory in fuzzy logics (and in Łukasiewicz logic in particular) was pioneered by Robin Giles in his paper [2] (dialogue games approach) and Daniele Mundici in his paper [4] (the Ulam games approach). Nowadays the game theory and fuzzy logics are interacting and a very fruitful way. We contribute to this work by focusing on the framework of evaluation games (exploring the model theoretical features of the logic—for details see paper [1]). Nice properties of Łukasiewicz logic allow us to give an intuitive motivation of the basic ideas of the game semantics.

Evaluation game for a formula in Łukasiewicz logic can be given a "gambling" interpretation—we can see it as a betting game with a constant stake, moves of which consist in a redistributing the stake between two players (we shall call it as in the classical evaluation games Eloise—the original verifier and Abelard—the original falsifier). In the beginning Eloise bets on the validity of a particular formula in a particular model—she picks up a number between 0 and 1 expressing the minimal amount of the total stake she is able to win. If the current formula φ has the form $\psi_1\ c\ \psi_2$, Eloise (or the current verifier) is betting on subformulas (the admissible bets are given by the rules for the connective c and by the bet on the original formula), Abelard (or the current falsifier) chooses the formula to proceed with. If c is negation, the game proceeds with the role switch and continues with the formula ψ_1 with the bet increased. The game ends if the current subformula is an atomic one. If the degree of validity of in the model in question is greater or equal than the current bet on Eloise wins, otherwise it is a win for Abelard. The (sub)games corresponding to a formula the bet on which is 0 are immediately won by Eloise (or the current verifier).

Formally the evaluation game for Łukasiewicz predicate logic is similar to the classical one (in the sense of Hintikka and Sandu [3]). The game is defined by a formula φ , a model $\mathbf{M}$ and an MV-chain $\mathbf{L}$ (representing degrees of validity of formulas). The positions of the game are labeled by a (sub)formula φ, an $\mathbf{L}$-evaluation e and a degree of validity l (an element of the MV-chain $\mathbf{L}$). The rules for general quantification, classical conjunction and disjunction turn out to be the same as in the classical game (the degree of the validity does not change in a move), the rules for strong conjunction and disjunction on the contrary do modify the degree of validity. Unlike the rules of the classical game the rules for the strong connectives involve an action of both Eloise and Abelard. Negation consists again in a role switch but also includes a change of the degree of validity. It can be proven that the Łukasiewicz logic is complete with respect to the corresponding evaluation game. (There is a one one correspondence between a standard Tarskian validity in a model and an existence of the winning strategy for Eloise in the corresponding game).

The game semantics is in some sense more general than the standard interpretation. In particular the standard interpretation has to confine on the so called safe models where all the suprema and infima (required by the interpretation of the existential and general quantifiers) exist. The requirement of safeness is a crucial point of the usual Tarskian interpretation, which can be partially avoided by the proposed game semantics. We introduce a notion of g-validity (in the sense of existence of a winning strategy for Eloise), which on the safe models coincides with the classical ones, and show that we have an extension of Łukasiewicz logic which is g-complete with respect to a broader class of models than the safe ones.

After defining the game semantics we introduce the notion of informational independence (as studied in IF logics, see [5])—we allow quantified variables to be independent of the quantifiers to which they are syntactically subordinated. As in the standard IF logics we obtain formulas lacking a truth value. In particular we can have a formula and a range of degrees of validity (e.g. a subinterval of [0,1]), such that neither Eloise nor Abelard have a winning strategy for the corresponding game. (In a sense we have a formula which is neither partially true nor false). An interesting feature of fuzzy informational independence is that there can be an asymmetry between the strategies of Eloise and Abelard. While Abelard keeps his winning strategies for a certain range of degrees of validity, Eloise can lose some range of the degrees of validity, where she can win.

References:

[1] P. Cintula, O. Majer: *Evaluation games for fuzzy logics*, to appear.

[2] R. Giles: *A non-classical logic for physics*, Studia Logica 33:399-417, 1974.

[3] J. Hintikka, G. Sandu: *Game-theoretical semantics*, in J. van Benthem and A. ter Meulen (eds), Handbook of Logic and Language, Amsterdam, Elsevier, 361-410, 1997.
[4] D. Mundici: *Ulam Games, Łukasiewicz Logic and AFC***-Algebras*, Fundamenta Informaticae 18:151–161, 1993.
[5] A. Pietarinen, G. Sandu: *Games in philosophical logic*, Nordic Journal of Philosophical Logic 4:143-173, 1999.