States on pseudo BCK-semilattices

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In the last years, there appeared a number of algebraic structures that are non-commutative generalizations of known algebras related to logic. In the logical context this means that the strong conjunction is not commutative and the implication splits into two ones. Accordingly, G. Georgescu and A. Iorgulescu introduced pseudo BCK-algebras as an extension of BCK-algebras:

A structure $(A, \leq, \rightarrow, \rightsquigarrow, 1)$, where $\leq$ is a binary relation on A, $\rightarrow$ and $\rightsquigarrow$ are binary operations on A and 1 is a distinguished element of A, is called a *pseudo BCK-algebra* (pedantically, a *reversed left pseudo BCK-algebra*) if it satisfies the following axioms, for all $x, y, z \in A$:

1. $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$, $x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$,
2. $x \leq (x \rightarrow y) \rightsquigarrow y$, $x \leq (x \rightsquigarrow y) \rightarrow y$,
3. $x \leq x$,
4. $x \leq 1$,
5. $x \leq y$ and $y \leq x$ imply $x = y$,
6. $x \leq y$ iff $x \rightarrow y = 1$ iff $x \rightsquigarrow y = 1$.

For every pseudo BCK-algebra, the relation $\leq$ is a partial order with 1 as a greatest element. We restrict our attention to those pseudo BCK-algebras which are bounded join-semilattices with respect to $\leq$:

A *bounded pseudo BCK-semilattice* is an algebra $(A, \vee, \rightarrow, \rightsquigarrow, 0, 1)$, where $(A, \vee)$ is a join-semilattice, $(A, \leq, \rightarrow, \rightsquigarrow, 1)$ is a pseudo BCK-algebra such that $x \leq y$ iff $x \vee y = y$ for all $x, y \in A$, and 0 is the least element of $(A, \leq)$.

Bounded pseudo BCK-semilattices can be viewed as a natural generalization of pseudo MV-algebras or, more generally, of bounded integral residuated lattices. States on pseudo MV-algebras, pseudo BL-algebras and certain bounded integral residuated lattices were studied by many authors (D. Mundici, B. Riečan, G. Georgescu, A. Dvurečenskij, J. Rachůnek, etc.). We introduce and study this concept also for bounded pseudo BCK-semilattices:

A *Bosbach state* of a bounded pseudo BCK-semilattice $(A, \vee, \rightarrow, \rightsquigarrow, 0, 1)$ is a mapping $m : A \rightarrow [0, 1]$ such that (for all $x, y \in A$)

1. $m(0) = 0$ and $m(1) = 1$,

2. $m(x) + m(x \to y) = m(y) + m(y \to x)$,

3. $m(x) + m(x \rightsquigarrow y) = m(y) + m(y \rightsquigarrow x)$.

For bounded pseudo BCK-semilattices satisfying the identity $(x \to 0) \rightsquigarrow 0 = (x \rightsquigarrow 0) \to 0$ we define another kind of states, the so-called *Riečan states*. In certain residuated lattices (in particular, pseudo MV-algebras), Bosbach and Riečan states agree, but this is not the case for pseudo BCK-semilattices.