On Decidability of T-norm-Based Equational Theories

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The aim of this work is to show that the universal theory R of real closed fields [1] is interpretable in the equational theory of LPi1/2-algebras [2,3,6,7,8], and viceversa. Since R enjoys quantifier elimination, we will obtain that the full theory of R is interpretable in LPi1/2. This will also yield that any function definable in R is definable in LPi1/2.

As a consequence of this construction we provide a description of (left-continuous) t-norms [5] definable in LPi1/2. In particular we can find a complete characterization of definable continuous t-norms.

Theorem A continuous t-norm is definable iff it can be represented as a finite ordinal sum.

This is due to the fact that since the set of idempotent elements of a t-norm is definable, by the properties of real closed fields it must be a Boolean combination of semialgebraic sets [1].

Negative results are also given for the definability of left continuous t-norms, i.e.: a definable left continuous t-norm cannot have a dense set or an infinite discrete set of discontinuities. However, many well-known left-continuous t-norms obtained by some construction methods [4] are definable in LPi1/2. Hence, we directly have the following theorem.

Theorem The class of definable (left-continuous) t-norms is closed under (finite) ordinal sum, rotation, annihilation and rotation-annihilation.

Now, let $*$ be a definable left-continuous t-norm, and $\to_*$ its residuum. Then $[0,1]_{\mathcal{A}^*} = ([0,1], *, \to_*, \wedge, \vee)$ is a commutative, bounded, integral, residuated lattice. Suppose that the class of $\mathcal{A}^*$-algebras is generated by $[0,1]_{\mathcal{A}^*}$.

Lemma Let $*$ be a definable left-continuous t-norm. Then in $[0,1]_{\mathrm{LPi1/2}}$ there is a definable structure isomorphic to $[0,1]_{\mathcal{A}^*}$.

The left-continuous t-norm $*$ is definable in LPi1/2, and so is its residuum, hence they have as corresponding LPi1/2-functions $*'$ and $\to_{*'}$. Then, we can define a translation $\circ$ from $[0,1]_{\mathcal{A}^*}$-terms into LPi1/2-terms so that

- $(x * y)^\circ$ is $x *' y$
- $(x \to_* y)^\circ$ is $x \to_{*'} y$.

By construction we obtain the following

Theorem Let $*$ be a left-continuous triangular norm definable in LPi1/2. Suppose that the equational class of $\mathcal{A}^*$-algebras is generated by the algebra $[0,1]_{\mathcal{A}^*}$ arising from the real unit interval. Then an equation holds in the class of $\mathcal{A}^*$-algebras iff its translation holds in the class of LPi1/2-algebras.

Given the previous theorem, the fact that the universal theory of real closed fields is decidable, and the mutual definability between R and LPi1/2, we immediately have the following result.

Theorem Let $*$ be a left-continuous triangular norm definable in LPi1/2. Suppose that the equational class of $\mathcal{A}^*$-algebras is generated by the algebra $[0,1]_{\mathcal{A}^*}$ arising from the real unit interval. Then the equational logic of $\mathcal{A}^*$-algebras is decidable.

References

[1] Bochnak, J., Coste, M. and Roy, M. *Real Algebraic Geometry*. Springer, Heidelberg, 1998.

[2] Cintula P. A note to the definition of the LPi1/2-algebras. *Soft Computing*, Vol. 9 (8), 575–578, 2005.

[3] Esteva F., Godo L. and Montagna F. The LPi1 and LPi1/2 logics: two complete fuzzy logics joining Łukasiewicz and product logic. *Archive for Mathematical Logic*, 40, 39–67, 2001.

[4] Jenei, S. How to construct left-continuous triangular norms: state of the art. *Fuzzy Sets Systems*, 143, 27–45, 2004.

[5] Klement E.P., Mesiar R. and Pap E. *Triangular Norms*. Kluwer Academic Publisher, Dordrecht, The Netherlands, 2000.

[6] Montagna F. An algebraic approach to propositional fuzzy logic. *Journal of Logic, Language, and Information*, 9, 91–124, 2000.

[7] Montagna F. Functorial representation theorems for MV_Δ algebras with additional operators. *Journal of Algebra*, 238, 99–125, 2001.

[8] Montagna F., Panti G. Adding structures to MV-algebras. *Journal of Pure and Applied Algebra*, 164, 365–387, 2001.