Cancellative Residuated Lattices and Lattice Ordered Groups

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A famous result by Mundici establishes a categorial equivalence between lattice ordered abelian groups with a strong unit and MV-algebras. Another classical result in Algebraic Logic establishes a translation of Intuitionistic Logic in S4 which can be expressed as a categorial equivalence between Heyting algebras and Boolean algebras with an interior operator. In this paper we try to relate somehow these results. A conucleous in a residuated lattice L is an interior operator on L whose fixed points constitute a submonoid of L. We will consider the catogory LG_{cn} of ℓ-groups with a conucleous such that the image of the conucleous generates the whole group (of course morphisms are the conucleous preserving homomorphisms). On the other side, we will consider the category ORL of cancellative residuated lattices C such that for all x,y the sets Cx and Cy have non-empty intersection. These cancellative residuated lattices are said to be right-reversible or Orey residuated lattices. We do not know if they constitute a variety, but they contain interesting varieties, namely:

(a) The variety of cancellative residuated lattices C such that for every x, xC=Cx (these are axiomnatized by x(x}xy)=(xy/x)x=xy).

(b) The variety of commutative and cancellative residuated lattices.

The main result of this paper is a categorial equivalence between ORL and LG_{cn}. The result immediately extends to a categorical equivalence between commutative cancellative residuated lattices and lattice ordered abelian groups with a conucleous whose image generates the whole group. We also prove that the according to this equivalence equational conditions about Orey residuated lattices correspond to equational properties of the conucleous. for instance, divisibility corresponds to the fact that the image of the conucleous is downwards closed, and in the commutative case representability corresponds to the equation $\sigma(x \vee y) = \sigma(x) \vee \sigma(y)$. Finally, as a corollary we prove a categorial equivalence between product algebras with a conucleous and ΠMTL-algebras.