Modal operators on bounded commutative residuated ℓ-monoids

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Commutative residuated lattice ordered monoids ($R\ell$-monoids) are duals to commutative $DR\ell$-monoids which were introduced by Swamy in the 60s as a common generalization of Abelian lattice ordered groups and Brouwerian algebras. Also algebras of logics behind fuzzy reasoning can be considered as particular cases of bounded commutative $R\ell$- monoids defined as follows:

A *bounded commutative $R\ell$-monoid* is an algebra $M = (M; \odot, \vee, \wedge, \rightarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ satisfying the following conditions.

(i) $(M; \odot, 1)$ is a commutative monoid.
(ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice.
(iii) $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$, for any x, y, $z \in M$.
(iv) $x \odot (x \rightarrow y) = x \wedge y$, for any x, $y \in M$.

Namely from this point of view, MV-algebras, an algebraic counterpart of the Łukasiewicz infinite-valued propositional logic, are precisely bounded commutative $R\ell$-monoids satisfying the double negation law. Further, BL-algebras, an algebraic semantics of the Hájek basic fuzzy logic, are just bounded commutative $R\ell$-monoids isomorphic to subdirect products of linearly ordered commutative $R\ell$-monoids. Heyting algebras (duals to Brouwerian algebras), i.e. algebras of intuitionistic logic, are characterized as bounded commutative $R\ell$-monoids with idempotent multiplication.

Modal operators (special cases of closure operators) on Heyting algebras were introduced and studied by Macnab. Analogously, modal operators on MV-algebras were introduced recently.

In this talk we will define modal operators and strong modal operators for arbitrary bounded commutative $R\ell$-monoids.

Let M be an $R\ell$-monoid. A mapping $f: M \longrightarrow M$ is called a *modal operator* on M if, for any x, $y \in M$,

1. $x \leq f(x)$;
2. $f(f(x)) = f(x)$;
3. $f(x \odot y) = f(x) \odot f(y)$.

If, moreover, for any x, $y \in M$,

4. $f(x \oplus y) = f(x \oplus f(y))$,

then f is called a *strong modal operator* on M.

We will describe their properties and we will give the criterion for a modal operator on a bounded commutative $R\ell$-monoid. We will also deal with their properties in the class of normal $R\ell$-monoids in particular. This class is considerably wide because every BL-algebra and every Heyting algebra is normal. Further, we will show examples of modal operators on bounded commutative $R\ell$-monoids constructed by elements having the complement.