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LOCAL ADDITIVE MEASURES ON PERFECT FUZZY STRUCTURES

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In this paper we define and study the local states on perfect pseudo-MV algebras. The main result consists of proving that there is a one to one correspondence between the local states on strong perfect pseudo-MV algebras and the states on ℓ-groups. As a generalization of these states, we introduce the notion of a local additive measure on a perfect pseudo-MTL algebra and we prove that, in some conditions, a local additive measure can be extended to a Riečan state. It is given a necessary and sufficient condition for a local additive measure on a perfect pseudo-MTL algebra to be a Bosbach state.

A pseudo-MV algebra $(A, \oplus, ^-, ^\sim, 0, 1)$ is called *local* if it has a unique maximal ideal. A local pseudo-MV algebra A is called *perfect* if for any $x \in A$, $ord(x) < \infty$ implies $ord(x^-) = \infty$.
The intersection of all maximal ideals of a pseudo-MV algebra is denoted by $Rad(A)$ and it is called the *radical* of A.
A local pseudo-MV algebra A is perfect iff $A = Rad(A) \cup Rad(A)^*$.
We denote by $Id(a)$ the ideal generated by the element $a \in A$.
If A is a pseudo-MV algebra, we denote $D(A) = \{x \in A | ord(x) = \infty\}$ and $D(A)^* = \{x \in A | x \geq y^- \text{ for some } y \in D(A)\}$.
We also have $D(A)^* = \{x \in A | x \geq y^\sim \text{ for some } y \in D(A)\}$.

Definition

If A is a perfect pseudo-MV algebra, then a *local state* on A is a function $s : Rad(A) \to \mathbb{R}_+$ satisfying the conditions:
(ls_1) $s(0) = 0$;
(ls_2) $s(x \oplus y) = s(x) + s(y)$ for all $x, y \in Rad(A)$.
If $a \in A$ such that $Rad(A) = Id(a)$ then a local state s on A is *normalized* if $s(a) = 1$. A local state s is *faithful* if $s(x) \neq 0$ for all $x \in Rad(A)$, $x \neq 0$.

Definition

A perfect pseudo-MV algebra A is called *strong perfect* iff $x^- = x^\sim$ for all $x \in A$.

Proposition

Let A be a perfect pseudo-MV algebra and s a local state on A. Then, for all $x, y \in Rad(A)$ the following hold:
(1) if $x \leq y$ then $s(y) - s(x) = s(y * x^-) = s(x^\sim * y)$;
(2) $s(x \vee y) + s(x \wedge y) = s(x) + s(y)$;
(3) $s(x \oplus y) + s(y * x) = s(x) + s(y)$.

Theorem

If A is a strong perfect pseudo-MV algebra and G an ℓ-group such that $G = \mathcal{D}(A)$, then there is a one to one correspondence between the local states on A and the states on G. Under this correspondence, if $Rad(A) = Id(a)$ and $u = [a, 0]$ is a strong unit of G, the normalized local states on A are mapped onto normalized states on G.

If $(A, \wedge, \vee, *, \rightarrow, \sim>, 0, 1)$ is a pseudo-MTL algebra, we will denote:
$D(A) = \{x \in A \mid ord(x) = \infty\}$ and $D(A)^* = \{x \in A \mid ord(x) < \infty\}$.
The intersection of all maximal filters of a pseudo-MTL algebra A is called the *radical* of A and it is denoted by $Rad(A)$. The pseudo-MTL algebra A is called *local* if it has a unique maximal filter and in this case $Rad(A) = D(A)$. A is called *perfect* if it is good and for any $x \in A$, $ord(x) < \infty$ iff $ord(x^-) = \infty$ iff $ord(x^\sim) = \infty$ (see [1]).
If A is a perfect pseudo-MTL algebra, then $A = Rad(A) \cup Rad(A)^*$.
We define a binary operation $\oplus$ on a pseudo-MTL algebra A by
$x \oplus y := (y^\sim * x^\sim)^-$ for all $x, y \in A$.
If A is a good pseudo-MTL algebra we say that two elements $x, y \in A$ are *orthogonals*, denoted $x \perp y$, if $x^{-\sim} \le y^\sim$.

Lemma

Let A be a perfect pseudo-MTL algebra.
(1) if $x, y \in Rad(A)^*$, then x and y are orthogonals;
(2) if $x, y \in Rad(A)$, then x and y are not orthogonals.

Let A be a pseudo-MTL algebra and $X \subseteq A \setminus \{0\}$. An element $x \in A$ is called *X-zero divisor* if there is $y_1, y_2 \in X$ sucht that $x * y_1 = y_2 * x = 0$. If 0 is the only $Rad(A)-$zero divisor of A, then A is called *relative free of zero elements*.

Definition

A *Bosbach state* on a pseudo-MTL algebra A is a function $s : A \rightarrow [0, 1]$ such that the following conditions hold for all $x, y \in A$:
(bs_1) $s(x) + s(x \rightarrow y) = s(y) + s(y \rightarrow x)$;
(bs_2) $s(x) + s(x \sim> y) = s(y) + s(y \sim> x)$;
(bs_3) $s(0) = 0$ and $s(1) = 1$.

If x and y are two orthogonal elements of a pseudo-MTL algebra A, then we define a partial operation " + " on A by $x + y := x \oplus y$.

Definition

Let A be a good pseudo-MTL algebra. A *Riečan state* or *additive measure* on A is a function $s : A \longrightarrow [0, 1]$ such that the following conditions hold for all $x, y \in A$:
(rs_1) if $x \perp y$, then $s(x + y) = s(x) + s(y)$;
(rs_2) $s(1) = 1$.

It was proved in [2] that every Bosbach state on a pseudo-MTL algebra A is a Riečan state, but the converse is not true. Moreover, we proved in [3] that every perfect pseudo-MTL algebra admits at least a Bosbach state.
According to the previous Lemma, for all $x, y \in Rad(A)^*$ we have $x \perp y$, so the operation $+$ is defined for all elements of $Rad(A)^*$.

Definition

If A is a perfect pseudo-MTL algebra, then a *local additive measure* on A is a function $s : Rad(A)^* \longrightarrow [0, 1]$ satisfying the conditions:
(ls_1) $s(x + y) = s(x) + s(y)$ for all $x, y \in Rad(A)^*$;
(ls_2) $s(0) = 0$.

Examples

Let A be a perfect pseudo-MTL algebra. Then:
(1) The function $s : Rad(A)^* \longrightarrow [0, 1]$, $s(x) = 0$ for all $x \in Rad(A)^*$ is a local additive measure on A;

(2) If S is a Riečan state on A, then $s = S/Rad(A)^*$ is a local additive measure on A.

According to the previous Lemma it follows that the function s is well defined, i. e. $x \oplus y \in Rad(A)^*$ for all $x, y \in Rad(A)^*$.

Proposition

If s is a local additive measure on the perfect pseudo-MTL algebra A, then the following hold for all $x, y \in Rad(A)^*$:
(1) $s(x^{-\sim}) = s(x)$;
(2) $s(x) + s(y^{--}) = s((y^- * x^\sim)^-)$ and $s(x) + s(y^{\sim\sim}) = s((y^\sim * x^-)^\sim)$;
(3) $s(x^{--}) + s((x^\sim * x^-)^\sim) = s(x^{\sim\sim}) + s((x^- * x^\sim)^-)$;
(4) $s(x) \leq s((x^- * x^\sim)^-)$ and $s(x) \leq s((x^\sim * x^-)^\sim)$.

If s is a local additive measure on the perfect pseudo-MTL algebra A, then we define the function $s^* : Rad(A) \longrightarrow [0,1]$ by $s^*(x) = 1 - s(x^- \oplus x^\sim)$ for all $x \in Rad(A)$.

Proposition

If s is a local additive measure on the perfect pseudo-MTL algebra A, then the following hold for all $x, y \in Rad(A)$:
(1) $s^*(1) = 1$;
(2) $s^*(x^{-\sim}) = s^*(x)$;
(3) $s^*(x \oplus y) = 1 - [s(y^- * x^-) + s(y^\sim * x^\sim)]$;
(4) $1 + s^*(x) \leq s^*(x^{--}) + s^*(x^{\sim\sim})$;
(5) $s^*(x \oplus y) = s^*(x) + s^*(y)$ iff $s(y^- * x^-) = s(y^\sim * x^\sim) = 0$;
(6) $min\{s(x^-), s(x^\sim)\} \leq 1/2$.

Theorem (Extension theorem)

Let A be a perfect pseudo-MTL algebra relative free of zero divisors. Then every local additive measure on A can be extended to a Riečan state on A.

Theorem

Let A be a perfect pseudo-MTL algebra relative free of zero divisors. The extension of a local additive measure s on A is a Bosbach state on A if and only if $s(x) = 0$ for all $x \in Rad(A)^*$.

References

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