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RELATIVE MV-ALGEBRAS AND RELATIVE HOMOMORPHISMS

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Several times it happens that given an MV-algebra A, special subsets of A, which are MV-algebras but not MV-subalgebras of A, are considered, and that they help in getting information about A. Indeed the same happens in the theory of Boolean Algebras, where are considered the so-called *relative* algebras, see [9]. We recall that Sikorski [10] and Tarski [11] proved the following generalization of the Cantor-Bernstein theorem: For any two σ-complete Boolean algebra A and B and elements $a \in A$ and $b \in B$, if B is isomorphic to the interval $[0, a] \subseteq A$ and A is isomorphic to $[0, b] \subseteq B$, then A and B are isomorphic. It can be seen, then, that subsets of Boolean algebras which are Boolean algebras play a role. Generalizations to MV-algebras, of the above mentioned theorems, say Cantor-Bernstein type theorems, involve a similar structure in MV-algebraic setting, i.e. the structure of *interval MV-algebra* subset of an MV-algebra, see for example [4], [5], [6].

We recall that in decomposing an MV-algebra A as a direct product sometime are considered MV-algebras whose underlying set is a subset $(b]$ of A, where b is an idempotent element of A and $(b]$ is the principal ideal of A generated by b. The MV-algebraic structure on $(b]$ is defined in a canonical way, see [2] where a decomposition of complete MV-algebras is proved. It is worth to observe that in the MV-algebra A the MV-algebraic structure over $(0, b]$ is defined with the help of the map $h_b : A \to A$, just setting $h_b(x) = b \wedge x$ and $\neg_b x = b \wedge \neg x$. Then $((b], \oplus, \neg_b, 0)$ is an MV-algebra and h_b is a homomorphism of A onto $(b]$. Also it can be trivially observed a property of h_b, actually the identity map $\delta : h_b(A) \to A$ is such that $h_b \circ \delta = ID_{h_b(A)}$. Such a trivial property assumes more significance in a wider categorical context. In [1] the authors defined an MV-algebraic structure on the interval $[0, a]$ of a given MV-algebra A, with $a \in A \setminus \{0\}$. Denoted such algebra A_a, they called it a *pseudo-subalgebra* of A. Then, it turns out that every MV-algebra A' is a pseudo-subalgebra of some perfect MV-algebra A. An analogous construction was presented in [7] and [8] where is defined a structure of MV-algebra over the interval $[a, b]$ of an arbitrary MV-algebra A, with $a, b \in A$.

Here we generalize the aforementioned constructions showing that one can uniformly define subsets of A (not necessarily intervals) which are MV-algebras; these MV-algebras are called Relative MV-subalgebras. The existence of Relative MV-subalgebras pushes us to consider a new category of MV-algebras having as objects still MV-algebras, but different morphisms, morphisms which are more general than the MV-homomorphisms. Following this line we can define an intermediate category, still having MV-algebras as objects and, as morphisms between MV-algebras A and B, maps which are not MV-homomorphisms but, roughly speaking, preserving MV-algebras which are intervals in A and in B, respectively. This allows to express, for example, the Cantor- Bernstein type theorem, for Boolean Algebras, above mentioned referring to Sikorski and Tarski, in categorical terms inside this new category.

As we will show the new class of morphisms, between MV-algebras, helps in describing a hidden relationship between the cyclic free MV-algebras of locally finite subvarieties generated by all finite chains S_i, with $i \leqslant n$ and the one-generated free MV-algebra in the variety of all MV-algebras.

Actually we show that:

(1) up to isomorphism, every one-generated free MV_p-algebra is a relative subalgebra of the cyclic free MV-algebra $F(1)$, for any integer positive p;
(2) up to isomorphism, the set of one-generated free MV_p-algebras, p varying in the set of all positive integer numbers, forms a directed system in the category of relative MV-algebras;
(3) up to isomorphism, each one-generated free MV_p-algebra is a retractive subalgebra of $F(1)$, in the category of relative MV-algebras.
(4) there is a family $\mathcal{D} = \{D_p\}_{p \in N}$ of finite sequences of elements of $Q \cap [0, 1]$ (subFarey sequences), such that each element $D_p \in \mathcal{D}$ allows us to cut out a relative subalgebra of $F(1)$, which isomorphic to $F_p(1)$.

References

[1] L.P. Belluce, A. Di Nola, *Yosida Type Representation for Perfect MV-algebras*, Math. Log. Quart. **42** (1996) 551-563
[2] R.L.O. Cignoli, I.M.L. D'ottaviano, D. Mundici, *Algebraic Foundations of Many-valued Reasoning*, Kluwer, Dordrecht. (Trends in Logic, Studia Logica Library) (2000).
[3] A.Di Nola, R.Grigolia, G. Panti, *Finitely Generated Free MV-algebras and Their Automorphism Groups*, Studia Logica 61(1) (1998) 65-78
[4] A.De Simone, D.Mundici, M.Navara, *A Cantor-Bernstein theorem for σ-complete MV-algebras*, Czechoslovak Math.J, 53 (128), (2003), 437-447.
[5] A.Di Nola, M. Navara *Cantor-Bernstein Property for MV-Algebras*, Algebraic and Prooftheoretic Aspects of Non-classical Logics (2006) 107-118
[6] J.Jakubik *Cantor-Bernstein theorem for MV-algebras*, Czechoslovak Math.J, 49 (124), (1999), 517-526.
[7] F. Lacava, *Sulla struttura delle L-algebre* , Atti Accad. Naz. Lincei Rend. Cl. Sci. Mat. Natur. (8), 5 (1979), 275-281
[8] F. Lacava, *Una caratterizzazione delle L-algebre complete, prive di atomi*, Bollettino U.M.I. (7) 9-A (1995), 609-618
[9] J.D. Monk *Handbook of Boolean Algebras* Basel North Holland (1989)
[10] R. Sikorski, *A generalization of a theorem of Banach and Cantor-Bernstein*, Colloquium Math., **1**, 140-144, 242 (1948)
[11] A. Tarski, *Cardinal Algebras.* Oxford University Press, New York (1949)

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