ON STATES IN MV-ALGEBRAS AND THEIR APPLICATIONS

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In the frames of quantum structures, states are a very important notion that model probability on an algebraic structure. Already G. Boole observed that to calculate probability of event of the structure M, it is important to know only which two events A and B we can add such that if $C = A + B$, and P is a probability, then $P(A + B) = P(A) + P(B)$; and the operation $+$ is a partial one on M. If M is a Boolean algebra, then $A+B := A\cup B$ whenever $A\cap B = \emptyset$ or equivalently, $A \le B'$. Such two events A and B are said to be mutually excluding or summable, [DvPu].

Therefore, the **state** or **finitely additive state** on an algebraic structure $(M; +, ', 0, 1)$ is any mapping $s : M \to [0, 1]$ such that (i) $s(1) = 1$, and (ii) $s(a + b) = s(a) + s(b)$ whenever $a + b$ is defined in M. The unary operation $'$ is an orthogonal complement or a negation.

The basic task is how to define a state on an algebraic structure like $(M; \oplus, \odot, ^*, 0, 1)$ or $(M; \oplus, \odot, ^-, ^\sim, 0, 1)$, that is, how to derive a partial operation $+$ from the $\oplus, \odot$?

In more complicated structures, like orthomodular lattices and effect algebras, the most important example is the system $\mathcal{L}(H)$ of all closed subspaces of a Hilbert space H or the system of all Hermitian operators $\mathcal{E}(H)$ that are between the zero operator and the identity operator. Here the σ-additive states are of the form

$$s_\phi(M) = (P_M\phi, \phi), M \in \mathcal{L}(H),\ \phi \in H,$$
$$s(M) = \sum_i \lambda_i s_\phi(M) = \mathrm{tr}(TP_M),\ M \in \mathcal{L}(H).$$

Gleason theorem, 1957, $3 \le \dim H \le \aleph_0$, [Dvu].

If s is a finitely additive on $L(H)$, Aarnes [Dvu],

$$s = \lambda s_1 + (1 - \lambda)s_2$$

where s_1 is a σ-additive state on $\mathcal{L}(H)$ and s_2 a finitely additive state vanishing on each finite-dimensional subspace of H.

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On the other hand, each Boolean algebra has plenty finitely additive states, but there is an example of Boolean σ-algebra that has no σ-additive states. There are plenty of OML's or OMP's that are stateless.

An **effect algebra**, Foulis and Bennett (D-poset Kôpka and Chovanec) [DvPu], is a partial algebra $(M; \oplus, 0, 1)$ such that (i) $+$ is associative and commutative, (ii) for each $a \in M$ there is a unique $a' \in M$ such that $a + a' = 1$, and (iii) if $a + 1$ is defined in M then $a = 0$.

Example: Let $(G; +, \leq)$ be a po-group, and $u > 0$. Set $\Gamma(G, u) = [0, u]$, and let $+$ be the restriction of the group addition to $\Gamma(G, u)$. Then $\Gamma(G, u)$ is an effect algebra (called an interval effect algebra). Each such an effect algebra admits at least one state. For example, if $\mathcal{B}(H)$ is the system of all Hermitian operators on H, then $\mathcal{E}(H) = \Gamma(\mathcal{B}(H), I)$.

The **Riesz decomposition property**: If $c \leq a + b$ then there are $a_1, b_1 \in M$ such that $a_1 \leq a$, $b_1 \leq b$ and $c = a_1 + b_1$. This is equivalent if $a_1 + a_2 = b_1 + b_2$, there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in M$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$, and $b_2 = c_{21} + c_{22}$.

Ravindran showed that if an effect algebra M satisfies RDP, then there is a unique unital interpolation po-group (G, u) such that $M = \Gamma(G, u)$, [DvPu].

If $(M; \oplus, \odot, ^*, 0, 1)$ is an MV-algebra, then defining a partial operation $+$ on M via $a + b$ is defined iff $a \odot b = 0$ (equivalently, $a \leq b^*$), and then we set $a + b = a \oplus b$. Then $(M; +, 0, 1)$ is an effect algebra with RDP.

A state s is said to be **extremal** if from $s = \lambda s_1 + (1-\lambda)s_2$, $0 < \lambda < 1$, we have $s = s_1 = s_2$. Let $\mathcal{S}(M)$ and $\mathcal{S}_\partial(M)$ be the system of all states, respectively extremal states, on M. If $\mathcal{S}(M) \neq \emptyset$, then, due to the Krein-Milman theorem

$$\mathcal{S}(M) = (\mathrm{ConHull}(\mathcal{S}_\partial(M)))^{-w}.$$

If s is state on EA M, then

$$\mathrm{Ker}(s) = \{a \in M : \ s(a) = 0\}$$

is an ideal of M.

If M is an MV-algebra, then a state s is extremal iff $\mathrm{Ker}(s)$ is a maximal ideal on M. There is a one-to-one correspondence between extremal states and maximal ideals on M via $s \leftrightarrow \mathrm{Ker}(s)$, s is extremal iff $s(a \oplus b) = \min(s(a) + s(b), 1)$. $\mathcal{S}_\partial(M)$ is a compact Hausdorff topological space. If M is only an EA, this is not true, in general. The situation with GMV-algebras or pseudo effect algebras is similar but more complicated [Dvu3].

A **tribe** is a system $\mathcal{T}$ of fuzzy sets on Ω such that (i) $1_\Omega \in \mathcal{T}$, (ii) if $f \in \mathcal{T}$, then $1 - f \in \mathcal{T}$, and (iii) if a sequence $\{f_n\}$ is taken from $\mathcal{T}$, then $\bigoplus_n f_n := \min(\sum_n f_n, 1) \in \mathcal{T}$. For any family $\mathcal{F}$ of fuzzy sets on Ω, there is a minimal tribe generated by $\mathcal{F}$.

Mundici [Mun], Dvurečenskij [Dvu1], Barbieri and Barbieri [BaWe]:

Theorem 1 (Loomis-Sikorski)**.** *Every σ-complete MV-algebra M is a σ-epimorphic image of a tribe. This tribe can be choose to be generated by extremal states on M.*

Buhagiar, Chetcuti, Dvurečenskij, [BCD]:

Theorem 2. *Every σ-complete EA M with RDP is a σ-epimorphic image of a σ-complete EA of fuzzy sets with pointwise defined operation satisfying RDP.*

Another approach is via state MV-algebras, Flaminio and Montagna, [FlMo]. We define a more narrow notion: an **extremal state MV-algebra** is a system $(M; \oplus, \odot, ^*, \sigma, 0, 1)$ where σ is a homomorphism of the MV-algebra M into M such that $\sigma^2 = \sigma$. We describe all subdirectly irreducible elements of the variety of extremal state MV-algebras (Di Nola and Dvurečenskij, [DiDv]):

Theorem 3. *(M, σ) is subdirectly irreducible iff either M is a subdirectly irreducible MV-algebra or $M = A \times B$, where A is a chain MV-algebra, B a subdirectly irreducible MV-algebra, and $\sigma(a, b) = (a, h(a))$, $(a, b) \in A \times B$, where $h: A \to B$ is an MV-homomorphism.*

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