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SMV-ALGEBRAS AND PROBABILISTIC KRIPKE MODELS

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States on MV-algebras have been introduced by Mundici in [Mu] as averaging processes for formulas in Łukasiewicz logic.

In order to treat states in a logical framework, Flaminio and Godo introduce in [FG07] the logics $FP(\text{Ł}_n, \text{Ł})$ and $FP(\text{Ł}, \text{Ł})$. The latter is obtained by adding a unary modality Pr to the language of Łukasiewicz logic (cf [H98]) and modal axioms suggested by the following semantic interpretation: the probability of an event A is interpreted as the truth value of the modal formula $\Pr(A)$.

Well-founded formulas of $FP(\text{Ł}, \text{Ł})$ are constituted of all the formulas of Łukasiewicz logic (those are the non-modal formulas), and the class of modal formulas so defined: for each non-modal formula A, $\Pr(A)$ is a modal formula, the truth constant $\overline{0}$ is modal, finally these formulas are combined by means of the Łukasiewicz connectives. This means that, for instance, neither $\Pr(A \to \Pr(B))$ nor $B \oplus \Pr(A)$ (A and B being Pr-free) are well-founded formulas. Using those modal fuzzy logics one can treat probability over many-valued events.

A *probabilistic Kripke model* for the logic $FP(\text{Ł}, \text{Ł})$ is a pair $K = (W, \mu)$ where W is a set of valuations of Łukasiewicz formulas in $[0, 1]$ and $\mu : W \to [0, 1]$ satisfies the condition: $\sum_{w \in W} \mu(w) = 1$. Elements of W are also called *nodes* or *possible worlds*.

Given a Kripke model $K = (W, \mu)$ and a formula A of $FP(\text{Ł}, \text{Ł})$, the truth value $\|A\|_{K,w}$ of A in K at the node w is inductively defined as follows:

- If A does not contain any occurrence of the modality Pr, then $\|A\|_{K,w} = w(A)$,
- If A is in the form $\Pr(B)$, then $\|A\|_{K,w} = \sum_{w \in W} w(\psi) \cdot \mu(w)$.
- Łukasiewicz connectives are treated truth-functionally as usual.

A natural expectation is that $FP(\text{Ł}, \text{Ł})$ may be complete with respect to probabilistic Kripke models.

As they are, the $FP(\text{Ł}_n, \text{Ł})$ and $FP(\text{Ł}, \text{Ł})$ logics are not algebraizable in the sense of Blok-Pigozzi. Recall in fact that $\Pr(A)$ is a well-founded formula only if A is a non-modal formula (and hence A does not contain any occurrence of Pr), and a formula of the form $B \oplus C$ is well-founded whenever B is modal iff C is modal. Therefore the algebraic counterpart of the operator Pr is a partial operation but not an operation.

As to provide an algebraic approach to states, in [FM07] we introduced the variety of SMV-algebras. An SMV-algebra is a system $A = (\mathbf{A}, \oplus, \neg, \sigma, 0, 1)$, where the reduct $(\mathbf{A}, \oplus, \neg, 0, 1)$ is an MV-algebra, and $\sigma : \mathbf{A} \to \mathbf{A}$ satisfies the following equations for each $x, y \in \mathbf{A}$:

- $\sigma(0) = 0$
- $\sigma(\neg x) = \neg\sigma(x)$
- $\sigma(\sigma(x) \oplus \sigma(y)) = \sigma(x) \oplus \sigma(y)$

- $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \ominus (x \odot y))$

where $x \odot y$ stands for $\neg(\neg x \oplus \neg y)$ and $x \ominus y$ stands for $\neg(\neg x \oplus y)$.

In [FM07,FM08] we show how, starting from an SMV-algebra A one can define a state s (in the sense of Mundici) on the MV-reduct of A. Furthermore, we use SMV-algebras to equationally characterize the coherence of a finite and rational-valued assessment over Łukasiewicz events:

Theorem 1 Let $\alpha : P(A_i) = n_i/m_i$ be a rational assessment over the Łukasiewicz formulas $A_1, \ldots, A_t$. Then the following are equivalent:

(a) α is coherent.
(b) The equations $\varepsilon_i : (m_i - 1)y_i = \neg y_i$, and $\delta_i : \sigma(A_i) = n_i y_i$ (for $i = 1, \ldots, t$, and the y_i's being fresh variables) are satisfied in some non-trivial SMV-algebra.

Let now $FP^+(Ł, Ł)$ be the logic obtained by extending $FP(Ł, Ł)$ by the following:

(a) the language is extended by the rules: (1) $\Pr(A)$ is a formula whenever A is a formula (without the restriction that A does not contain occurrence of Pr), and (2) if B and C are formulas (no matter if B and C are modal or not), then $B \oplus C$ is a formula (hence we also remove the restriction that B is modal iff so is C).
(b) the axioms of $FP(L, L)$ (cf [FG07]) are extended to the formulas of the new language.
(c) the axiom schema $\Pr(A) \leftrightarrow A$ is added, whenever A ranges over all formulas all of whose variables only occur under the scope of Pr (this axiom reflects the fact that such formulas represent real numbers which coincide with their probability).

Then the variety of SMV-algebras constitutes the natural algebraic semantic for $FP^+(L, L)$, and hence the $FP^+(L, L)$-formulas can be identified with SMV-terms.

An $FP^+(Ł, Ł)$-formula A is said 1-*satisfiable* if there exists a probabilistic Kripke model K such that $\|A\|_K = 1$. Our main result now reads as follows:

Theorem 2 Let A be a term in the language of SMV-algebras. Then the following are equivalent:

(i) There is a Kripke-model $K = (W, \mu)$ such that $\|A\|_K = 1$.
(ii) A is 1-satisfiable in an SMV-algebra $(\mathbf{A}, \sigma)$.

Using this result we also show that the problem of deciding whether an equation $A = 1$ holds in an SMV-algebra is PSPACE. The latter result, together with Theorem 1, provides a (new proof (cf [H07]) for the) PSPACE-containment for the problem of establishing the coherence of a rational assessment over Łukasiewicz formulas.

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