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CONTINUOUS T-NORM AS QUANTUM GATES: A PROBABILISTIC APPROACH

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Recently several authors paid attention to the extension of the classical notion of probability developed in the Boolean framerwork to more general algebraic structures. An important example is represented by the notion of states of MV-algebras [1,2,3]. On the other hand algebraic structures closely related to fuzzy logic constitute also an useful tool in investigating probabilistic aspects of systems associated to quantum computation [4]. The present work follows the latter stream.

A quantum system in a pure state is described by a unit vector in a Hilbert space. In the Dirac notation a pure state is denoted by $|\varphi\rangle$. A *quantum bit* or *qbit*, the fundamental concept of quantum computation is a pure state in the Hilbert space C^2. The standard orthonormal base $\{|0\rangle, |1\rangle\}$ of C^2 is called the *logical basis*. Thus a qbit $|\varphi\rangle$ may be written as a linear superposition of the basis vectors with complex coefficients $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ with $|c_0|^2 + |c_1|^2 = 1$. Quantum mechanics reads out the information content of a pure state via the Born rule. By these means, we consider the probability value assigned to a qbit as follows: $p(|\psi\rangle) = |c_1|^2$.

The quantum states of interest for quantum computation lie in the tensor product $\otimes^n C^2 = C^2 \otimes C^2 \otimes \ldots \otimes C^2$. The space $\otimes^n C$ is a 2^n-dimensional complex space. We choose a special basis for $\otimes^n C$ which is called the 2^n-*computational basis*. More precisely, it consists of the 2^n orthogonal states $|\iota\rangle$, $0 \leq \iota \leq 2^n$ where ι is in binary representation and ι can be seen as tensor product of states $|\iota\rangle = |\iota_1\rangle \otimes |\iota_2\rangle \otimes \ldots \otimes |\iota_n\rangle$ where $\iota_j \in \{0, 1\}$. A pure state $|\psi\rangle \in \otimes^n C$ is generally a superposition of the basis vectors $|\psi\rangle = \sum_{\iota=1}^{2^n} c_\iota |\iota\rangle$ with $\sum_{\iota=1}^{2^n} |c_\iota|^2 = 1$.

In general, a quantum system is not in a pure state. This may be attributed to the fact that the systems are not isolated from the rest of the universe, so it does not have a well defined pure state. We say that the system is in a *mixed state* which is described by a *density operator*. A density operator is a Hermitian (i.e $\rho^\dagger = \rho$) positive trace class operator on a 2^n-dimensional complex Hilbert space with trace $tr\rho = 1$. A pure state $|\psi\rangle$ can be represented by the operator $\rho = |\psi\rangle\langle\psi|$, where $\langle\psi| = (|\psi\rangle)^\dagger$. As a particular case, with each vector of the logical base of C^2 we consider the very important density operators $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ that represent,the truth-property and the falsity-property respectively.

One can represent an arbitrary density operator ρ for n-qbits in terms of tensor products of the Pauli matrices:

$$\sigma_0 = I \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in the following way: $\rho = \frac{1}{2^n} \sum_{\mu_1 \ldots \mu_n} P_{\mu_1 \ldots \mu_n} (\sigma_{\mu_1} \otimes \ldots \otimes \sigma_{\mu_n})$ where $\mu_i \in \{0, x, y, z\}$ for each $i = 1 \ldots n$ and $|P_{\mu_1 \ldots \mu_n}| \leq 1$. We denote by $D(\otimes^n C^2)$ the set of all density operators of $\otimes^n C^2$. Moreover, we can identify, in each space $D(\otimes^n C^2)$, two special operators $P_0^{(n)} = \frac{1}{2^n} I^{n-1} \otimes P_0$ and $P_1^{(n)} = \frac{1}{2^n} I^{n-1} \otimes P_1$ that represent, in this framework, the truth-property and the falsity-property. By applying the Born rule, we obtain the probability value corresponding to the fact that the physical system in the state $\rho \in D(\otimes^n C^2)$ satisfies the truth-property $P_1^{(n)}$ as follows: $p(\rho) = Tr(P_1^{(n)} \rho)$. In the particular case that $\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ we obtain that $p(\rho) = |c_1|^2$.

In the simplest representation of the work of a quantum computer, the state of the system is a pure state and the operations are represented by unitary operators. These operations are identified with the *quantum gates*. But when the system is open because it is either coupled to an environment or is being subject to a measurement, in general its time evolution is irreversible. A general model of quantum computation which takes into account this situation is captured mathematically by means of quantum operations as quantum gates [5], acting on density operators.

In general, let M be a finite dimensional complex Hilbert space and $L(M)$ be the vectorial space of all linear operators on M. A *quantum operation* is a linear operator $E : L(N) \to L(M)$ representable as

$$E(\rho) = \sum_i A_i \rho A_i^\dagger$$

for some set of operators $\{A_i\}$ such that $\sum_i A_i^\dagger A_i = I$. It can be seen that quantum operations send density operators to density operators.

In the present work we provide a probabilistic version of the well known Stone Weierstrass theorem applied to quantum operations, which will allow us to regard any continuous t-norm on $[0, 1]$ as a quantum operation. More precisely, let f be a continuous t-norm. Then, for each $\epsilon > 0$, there exists $M > 0$ and a quantum operation E such that, for any pair of density operators σ_1, σ_2,

$$| \, Mp(E(\sigma_1 \otimes \sigma_2)) - f(p(\sigma_1), p(\sigma_2)) \, | \leq \epsilon$$

In this way any continuous t-norm can be regarded as a quantum gate.

References

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