ManyVal '08 - Applications of Topological Dualities to Measure Theory in Algebraic Many-Valued Logic, May 19–21, 2008, University of Milan, Milan, Italy

STATES ON BOLD ALGEBRAS: CATEGORICAL ASPECTS

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We study bold algebras and states on bold algebras in the context of transition from classical random events, random variables, and observables to their quantum and fuzzy generalizations. We show that D-posets of fuzzy sets form a suitable category ID in which some basic constructions of measure theory and probability theory are natural.

Motivation. Let $(X, \mathbb{A})$ and $(Y, \mathbb{B})$ be classical measurable spaces and let f be a measurable map of X into Y. It is known that f defines the (dual) preimage map f^d sending each B in $\mathbb{B}$ to its preimage (the set of all x in X such that $f(x)$ is in B) and the preimage map is a sequentially continuous Boolean homomorphism of $\mathbb{B}$ into $\mathbb{A}$. Further, if p is a probability measure on $\mathbb{A}$, then the composition of f^d and p is a probability measure p_f on $\mathbb{B}$. This sends probability measures on $\mathbb{A}$ to probability measures on $\mathbb{B}$. As particular cases we get classical notions of probability theory: a random variable f, its distribution p_f, and the observable f^d.

In the recently developed fuzzy (or operational) probability theory, see e.g. [GUDDER 1998], [BUGAJSKI 2001a], [BUGAJSKI 2001b], we start with a map T of the probability measures $P(\mathbb{A})$ on $\mathbb{A}$ into the probability measures $P(\mathbb{B})$ on $\mathbb{B}$ satisfying a natural measurability condition which guarantees the existence of a dual map T^d of all measurable functions $M(\mathbb{B})$ of Y into the closed unit interval I = [0,1] into all measurable functions $M(\mathbb{A})$ of X into I so that T^d is a sequentially continuous ID-morphism (cf. [FRIC 2005a]).

Bold algebras and Lukasiewicz tribes. Measurable functions into I can be considered as bold algebras. For bold algebras, an ID-morphism need not be a sequentially continuous MV-algebra homomorphism. The category BD, the objects of which are bold algebras and the morphisms of which are sequentially continuous MV-algebra homomorphisms, has been studied in [FRIC 2002]. Lukasiewicz tribes are absolutely sequentially closed bold algebras and form an epireflective subcategory of BD. In our talk we describe the relationship between bold algebras and Lukasiewicz tribes as subcategories of ID.

D-posets of fuzzy sets. D-posets (introduced in [KOPKA and CHOVANEC 1994]), equivalently effect algebras, generalize MV-algebras and D-posets of fuzzy sets generalize bold algebras (see DVURECENSKIJ and PULMANNOVA 2000]). Indeed, let A be a bold algebra of functions of X into I and let A^* be the set of all functios a^* from the states on A into I defined by $a^*(s) = s(a)$, where a belongs to A and s is a state on A. Then A^* is a D-poset of fuzzy sets which fails to be a bold algebra, but A and A^* are isomorphic D-posets. This leads to the category ID the objects of which are D-posets of fuzzy sets and the morphisms of which are sequentially continuous (with respect to the pointwise sequential convergence) D-homomorphisms. More information about ID can be found, e.g., in [PAPCO 2004], [PAPCO 2007], [FRIC 2005a], [FRIC 2005b]. Observe that the fields of sets can be considered as a full subcategory of ID and the sigma-additive probability measures are exactly ID-morphisms into I.

Main results. 1. We describe the transition from a bold algebra to the generated Lukasiewicz tribe in the realm of ID. In particular, bold algebras belong to a distinguished subcategory $STID$ of ID and we prove that Lukasiewicz tribes form an epireflective subcategory of $STID$. The epireflection is related to the extension of measures.

2. We outline a model of probability theory having both fuzzy and quantum qualities. Basic probability notions are in terms of ID and generalize their classical counterparts.

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Thanks. This work was supported by the Slovak Research and Development Agency under the contract No. APVV-0071-06, Slovak Academy of Sciences via the project Center of Excellence - Physics of Information, and VEGA 2/6088/26.

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