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TOWARDS COALITION GAMES ON MV-ALGEBRAS

TOMAS KROUPA

Coalition game theory originated from the work [10] of von Neumann and Morgenstern. Further development of this theory led to many generalizations such as games with infinitely many players [2] of Aumann and Shapley or Aubin's games with fuzzy coalitions [1,5]. A coalition game can be determined by a set of players, a set of coalitions that can be formed by the players, and a mapping assigning to every coalition its profit resulting from the cooperation among the players constituting the coalition. One of the basic question investigated in this context is that of the existence of a certain operator called value, which embodies the idea of a "fair" distribution of the total profit among the individual players.

The aim of this contribution is to make first steps towards the study of coalition games in the MV-algebraic framework. It will be shown that the apparatus of MV-algebras [6] is rich enough to deal with a very large class of coalition games including both games with infitely many players and games with fuzzy coalitions simultaneously. Let $\langle \mathcal{M}, \oplus, \odot, \neg, 0, 1\rangle$ be a semisimple MV-algebra. We assume that $\mathcal{M}$ is the set of plausible coalitions. By the representation theorem from [4] every $A \in \mathcal{M}$ can be identified with a unique $[0,1]$-valued function on some compact Hausdorff space X. In this context an element of X is called a *player* and every $A \in \mathcal{M}$ is called a *coalition.* Note that a player $x \in X$ is allowed to participate in a coalition $A \in \mathcal{M}$ only partially in a degree given by $A(x)$. A *game* is a real function v on $\mathcal{M}$ with $v(0) = 0$. For every $A \in \mathcal{M}$ the number $v(A)$ can be viewed as the worth or the profit of the coalition A that is ensured by the members of A by acting towards a common goal of the coalition A.

We say that a real function m on $\mathcal{M}$ is a *finitely additive measure* (cf. [5,8,3]) when $m(0) = 0$ and for every $A, B \in \mathcal{M}$ such that $A \odot B = 0$, we have $m(A \oplus B) = m(A) + m(B)$. Let $\mathcal{M}$ be a σ-complete MV-algebra. A *measure* on $\mathcal{M}$ is a finitely additive measure on $\mathcal{M}$ such that $m(A) = \lim m(A_n)$ whenever $A = \bigvee A_n$, where A_n is a nondecreasing sequence of elements of $\mathcal{M}$. By $\mathsf{FBV}_{\mathcal{M}}$ we denote the Banach algebra of all games on $\mathcal{M}$ which are of bounded variation in the sense of [5,Definition 15.3] and by $\mathsf{FBA}_{\mathcal{M}}$ we denote its closed linear subspace of all finitely additive measures on $\mathcal{M}$ belonging to $\mathsf{FBV}_{\mathcal{M}}$. If α is an MV-algebraic automorphism of $\mathcal{M}$, then it induces a linear mapping $\alpha_* : \mathsf{FBV}_{\mathcal{M}} \to \mathsf{FBV}_{\mathcal{M}} : v \mapsto \alpha_* v$ given by $\alpha_* v(A) = v(\alpha A)$, for every $A \in \mathcal{M}$. According to [7] every automorphism of a semisimple MV-algebra also gives rise to a homeomorphism of the compact Hausdorff space X of all players. In particular, this result enables to build the correspondence between the automorphisms of $\mathcal{M}$ and permutations in the sense of [5,18.1]. A linear subspace Q of $\mathsf{FBV}_{\mathcal{M}}$ is called *symmetric* if $\alpha_* v \in \mathsf{Q}$ for every $v \in \mathsf{Q}$ and every automorphism α of $mathcalM$, and a linear operator $\mathsf{Q} \to \mathsf{FBA}_{\mathcal{M}}$ is *positive* when it maps monotone games in Q to monotone finitely additive measures in $\mathsf{FBA}_{\mathcal{M}}$. With these definitions in mind we introduce the following notion capturing the concept of a "fair" distribution of worth in the game v, which involves some commonly accepted axioms of rationality (see [9,2,5]).

Definition. Let $\mathcal{M}$ be a semisimple MV-algebra and Q be a symmetric subspace of $\mathsf{FBV}_{\mathcal{M}}$. An *Aumann-Shapley value* on Q is a positive linear operator $\varphi : \mathsf{Q} \to \mathsf{FBA}_{\mathcal{M}} : v \mapsto \varphi v$, which satisfies the following conditions:

(1) *Symmetry:* if α is an automorphism of $\mathcal{M}$, then for every $v \in \mathsf{Q}$

$$\varphi(\alpha_* v) = \alpha_*(\varphi v).$$

(2) *Efficiency:* $\varphi v(1) = v(1)$, for every $v \in \mathsf{Q}$.

In general, Aumann-Shapley value need not exist on the whole space $\mathsf{FBV}_{\mathcal{M}}$ provided $\mathcal{M}$ is an arbitrary semisimple MV-algebra. In case that $\mathcal{M}$ is a σ-complete MV-algebra, however, the existence of Aumann-Shapley value can be proved for a large class of games. By $\mathsf{pFNA}_{\mathcal{M}}$ we denote the closed linear span of all natural powers of monotone nonatomic measures on $\mathcal{M}$ (cf. [5,Definition 16.2]). A *nonatomic vector measure* is a vector $\mathbf{m} = (m_1, \ldots, m_n)$ of nonatomic measures $m_1, \ldots, m_n$. The following result is based on Theorem 18.4 in [5].

Theorem. If $\mathcal{M}$ is a σ-complete MV-algebra, then there exists an Aumann-Shapley value φ on $\mathsf{pFNA}_{\mathcal{M}}$ such that φ is continuous and has norm 1. Let $\mathbf{m}$ be a nonatomic vector measure on $\mathcal{M}$ and f be a continuously differentiable function of n real variables on the range of $\mathbf{m}$ with $f(0) = 0$. Then we have for every $A \in \mathcal{M}$,

$$\varphi(f \circ \mathbf{m})(A) = \int_0^1 f_{\mathbf{m}(A)}(t\mathbf{m}(1))\, dt,$$

where the integral on the right-hand side above is Riemann and $f_{\mathbf{m}(A)}(t\mathbf{m}(1))$ is the derivative of f at $t\mathbf{m}(1)$ in the direction $\mathbf{m}(A)$.

The question of interest is the existence and the uniqueness of Aumann-Shapley value on classes of games on a semisimple MV-algebra, which is not necessarily σ-complete.

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Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic

E-mail address: kroupa@utia.cas.cz