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A NOTION OF INDEPENDENCE FOR PROBABILITY MV-ALGEBRAS

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An appropiate definition for the notion of *stochastically independent* σ-subalgebras is an open problem mentioned by Riečan and Mundici in [7]. In the following, we propose a concept of *independent structures* for probability MV-algebras, generalizing the classical theory for probabilities defined on Boolean algebras [1,3].

In our approach, a *probability MV-algebra* is a pair (A, s) where A is an MV-algebra and $s : A \to [0, 1]$ is a (finitely additive) state. A *σ-probability MV-algebra* is a probability MV-algebra (A, s) such that A is σ-complete and s is a σ-continuous state. A state s is called *extreme* if it is also an MV-algebra homomorphism into the standard MV-algebra $[0, 1]$.

For any two MV-algebras A and B, the tensor product $A \otimes_o B$ was defined in [2]. The main difference between $A \otimes_o B$ and the MV-algebraic tensor product defined by Mundici in [6] is that we no more assume that the bilinear functions involved are bimorphisms. As a consequence, the tensor product $A \otimes_o B$ is uniquely defined, up to an isomorphism, by the following universal property:

for any MV-algebra C and for any bilinear function $\beta : A \otimes_o B \to C$ such that $\beta(1, 1) = 1$, there exists a unique MV-algebra homomorphism $f : A \otimes_o B \to C$ with the property that $f(a \otimes_o b) = \beta(a, b)$ for any $a \in A$ and $b \in B$.

Our first result is the following.

Theorem 1. If (A, s_A) and (B, s_B) are probability MV-algebras then there exists a unique extreme state $s_{A,B} : A \otimes_o B \to [0, 1]$ such that $s_{A,B}(a \otimes_o b) = s_A(a) \cdot s_B(b)$ (where $\cdot$ is the real product) for all $a \in A$, $b \in B$.

The above result suggets the following definition.

Definition 2. Let (A, s_A), (B, s_B) and (T, s_T) be (σ-probability) probability MV-algebras. We say that (A, s_A) and (B, s_B) are (T, s_T)-*independent* if there exists a bilinear function $\beta : A \times B \to T$ satisfying $s_T(\beta(a, b)) = s_A(a) \cdot s_B(b)$ for all $a \in A$, $b \in B$.

It is easy to see that if A and B are Boolean subalgebras of a Boolean algebra T and $P : T \to [0, 1]$ is a boolean probability, then we get the usual concept: since $\beta(a, b) = a \wedge b$ is a bilinear map, A and B are called P-independent if $P(a \wedge b) = P(a) \cdot P(b)$ for any $a \in A$, $b \in B$.

From Theorem 1 and Definition 2 we get the following result.

Theorem 4. For any two probability MV-algebras (A, s_A) and (B, s_B) there exists a probability MV-algebra (T, s_T) and a bilinear function $\beta_{A,B} : A \times B \to T$ such that the following proprieties hold:
(a) (A, s_A) and (B, s_B) are (T, s_T)-*independent*,
(b) s_T is an extreme state.
If $\beta_{A,B} : A \times B \to T$ is the bilinear map which gives the independence, then
(c) $\beta_{A,B}(A \times B)$ generates T as an MV-algebra,

(d) for any probability MV-algebra (C, m) and for any bilinear function $\gamma : A \times B \to C$ such that m is an extreme state and $m(\gamma(a, b)) = s_A(a) \cdot s_B(b)$ for any $a \in A$, $b \in B$, there exists a unique MV-algebra homomorphism $f : T \to C$ such that $mf = s_T$ and $f(\beta_{A,B}(a, b)) = \gamma(a, b)$ for any $a \in A$, $b \in B$.

We remark that the structure T is $A \otimes_o B$, s_T is the probability given by Theorem 1 and $\beta_{A,B}(a, b) = a \otimes_o b$ for any $a \in A$, $b \in B$. By (d), the structure $((T, s_T), \beta_{A,B})$ is uniquely defined up to an isomorphism. The above theorem can be generalized to arbitrary families of probability MV-algebras, following the classical theory [1,3].

In order to obtain a similar result for σ-probability MV-algebras, we need a preliminary construction: the (metric) completion of an MV-algebra with respect to a state. We refer to [4] and [1] for the similar construction in lattice ordered groups and Boolean algebras.

Definition 5. If (A, s) is a probability MV-algebra, we define $\rho_s(a, b) = s(d(a, b))$ for any a, $b \in A$. It follows that ρ_s is a pseudo-metric on A. Moreover, s and the MV-algebra operations are uniformly continuous w.r.t ρ_s. Let $(A^\#, \rho_s^\#)$ be the pseudo-metric completion of (A, ρ_s) and $\Phi : A \to A^\#$ the natural map [5].

Proposition 6. If (A, s) is a probability MV-algebra and s is extreme, then the following properties hold:
(a) $A^\#$ is a σ-complete MV-algebra,
(b) there exists a σ-continuous state $s^\# : A^\# \to [0, 1]$ such that $s^\#\Phi = s$.

Let (A, s_A) and (B, s_B) be σ-probability MV-algebras, $T = A \otimes_o B$ and s_T is $s_{A,B}$ from Theorem 1. If we define $T^\#$ and $s_T^\#$ as above, then $(T^\#, s_T^\#)$ is a σ-probability MV-algebra by Proposition 6.

Theorem 7. Under the above hypothesis, the σ-probability MV-algebras (A, s_A) and (B, s_B) are $(T^\#, s_T^\#)$-independent.

As a consequence, for any two σ-probability MV-algebras (A, s_A) and (B, s_B) there exists a σ-probability MV-algebra (T, s_T) such that (A, s_A) and (B, s_B) are (T, s_T)-independent.

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