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Coauthors: Tommaso Flaminio (Universita di Siena) Lluis Godo (IIIA-CSIC)

# POSSIBILISTIC STATES: A LOGICAL FORMALIZATION

ENRICO MARCHIONI

## Introduction

Probability measures are without any doubt the main tool of modelling and reasoning under uncertainty. However, in the field of uncertain reasoning, a variety of formalisms has been developed to capture different notions of non-additive uncertainty (see e.g. [3)]. The most general notion of uncertainty measure is that of Sugeno measures [6], also called Plausibility measures by Halpern [3]. In its simplest form, given a Boolean algebra $U = (U, \wedge, \vee, \neg, \overline{0}^{\mathcal{U}}, \overline{1}^{\mathcal{U}})$, a Sugeno measure is a mapping $\mu : U \to [0,1]$ verifying $\mu(\overline{0}^{\mathcal{U}}) = 0$, $\mu(\overline{1}^{\mathcal{U}}) = 1$, and the monotonicity condition $\mu(x) \leq \mu(y)$ whenever $x \leq^{\mathcal{U}} y$, where $\leq^{\mathcal{U}}$ is the lattice order in $U$. Of course, probability measures are Sugeno measures, but Sugeno measures encompass a larger family of measures, like capacities, upper and lower probabilities, Dempster-Shafer plausibility and belief functions, or possibility and necessity measures.

Certainly, it makes sense to consider appropriate extensions of these classes of non-additive measures on more general algebraic structures than Boolean algebras, similarly to the well-known case of *states*, that generalize the classical notion of (finitely additive) probability measures on MV-algebras [4, 5].

In this work, we focus on the investigation and logical formalization of meaningful generalizations of possibility and necessity measures over MV-algebras. By analogy, we will refer to them as *possibilistic states*. Following the approach developed in [2] for a logical treatment of states on MV-algebras of fuzzy events, one of our aims in this paper is to deal with possibilistic states on MV-algebras of fuzzy events in a logical setting and show completeness results with respect to two classes of Kripke structures .

## Possibility and Necessity Measures

A possibility measure on a (finite) Boolean algebra $U = (U, \wedge, \vee, \neg, \overline{0}^{\mathcal{U}}, \overline{1}^{\mathcal{U}})$ is a Sugeno measure $\mu^*$ satisfiying the following $\vee$-decomposition property $\mu^*(u \vee v) = \max(\mu^*(u), \mu^*(v))$, while a necessity measure is a Sugeno measure $\mu_*$ satisfying the $\wedge$-decomposition $\mu_*(u \wedge v) = \min(\mu_*(u), \mu_*(v))$. Possibility and necessity measures are *dual* in the sense that if $\mu^*$ is a possibility measure then $\mu^*$ defined as $\mu_*(u) = 1 - \mu^*(\neg u)$ is a necessity measure and conversely. If $U$ is the power set of some set $X$, then any dual pair of measures $(\mu^*, \mu_*)$ on $U$ is induced by a so-called *possibility distribution* $\pi : X \to [0,1]$ in such a way that, for any $A \subseteq X$, $\mu^*(A) = \max\{\pi(x) \mid x \in A\}$ and $\mu_*(A) = \min\{1 - \pi(x) \mid x \notin A\}$.

When moving from the realm of Boolean algebras to the realm of MV-algebras, one can still define possibility and necessity measures on a MV-algebra $A$ as mappings $\mu : A \to [0,1]$ such that $\mu(\overline{0}) = 0$ and $\mu(\overline{1}) = 1$ and satisfying the same corresponding decomposition properties as above, where now $\vee$ and $\wedge$ refer to the induced lattice operations in the MV-algebra, i.e. $u \wedge y = u \otimes (\neg u \oplus v)$ and

$u \vee v = u \oplus (\neg u \otimes v)$. We will call *possibilistic state* on a MV-algebra $A$ any pair $(\mu_*, \mu^*)$ where $\mu_*$ is a necessity measure over $A$ and $\mu^*$ is its dual possibility measure, i.e. defined by putting $\mu^*(u) = 1 - \mu_*(\neg u)$. Since $\mu^*$ is fully determined by $\mu^*$ (and vice-versa), sometimes we will also refer to only $\mu_*$ as possibilistic state.

Within a set-theoretical framework, there is already a lot of literature on particular extensions of possibilistic measures on (classical) sets to fuzzy sets (see e.g. [1]). Since the class of fuzzy sets over a given domain can be equipped with an MV-algebra structure (with the point-wise extensions of the operations in the standard MV-algebra on $[0,1]$), we will take advantage of discussions, proposed definitions and properties for our purposes. Indeed, contrary to what happens with Boolean algebras, a possibility distribution $\pi$ on a set $X$ does not uniquely characterize a pair of possibility and necessity measures on the MV-algebra $F(X) = \{A : X \to [0,1]\}$ of fuzzy subsets of $X$ (it is actually a Lukasiewicz tribe). Several proposals can be found on the literature. Here we will adopt the following ones, strongly relying on the standard MV operations: given $\pi : X \to [0,1]$, for any $A \in F(X)$ we define: $\Pi_\pi(A) = \sup_{x \in X} \pi(x) \otimes A(x)$ and $N_\pi(A) = \inf_{x \in X} \pi(x) \Rightarrow A(x)$ where $\otimes$ and $\Rightarrow$ are respectively the Lukasiewicz strong conjunction and implication on the standard MV-algebra. This definition of $(\Pi_\pi, N_\pi)$ yields a particular kind of possibilistic state on $F(X)$, extending the classical definition of possibility and necessity measures for crisp sets, and moreover it can be easily characterized in the following terms (we assume $X$ to be finite).

**Proposition**. A mapping $\mu : F(X) \to [0,1]$ satisfies: (i) $\mu(\emptyset) = 0$, $\mu(X) = 1$, (ii) $\mu(A \wedge B) = \min(\mu(A), \mu(B))$, and (iii) $\mu(A \oplus \overline{r}) = r \oplus \mu(A)$ for all $r \in [0,1]$, iff there exists $\pi : X \to [0,1]$ such that $\mu(A) = N_\pi(A) = 1 - \Pi_\pi(\neg A)$. Here $\overline{r}$ denotes the constant fuzzy set of value $r$.

## Logical Formalization

Based on the above and following [2], we define a modal-fuzzy logic $FN(Ł_n^+, RPL)$ based on the Rational Pavelka logic RPL, and on the $(n+1)$-valued Lukasiewicz logic expanded with the truth-constant $\overline{1/n}$, which will be dentoed as $Ł_n^+$. Formulas of $FN(Ł_n^+, RPL)$ split into two classes: (i) the set $Fm(V)$ of non-modal formulas $\varphi, \psi \ldots$, which are formulas of $Ł_n^+$ built from set of propositional variables $V = \{p_1, p_2, \ldots\}$ and the truth-constant $\overline{1/n}$; (ii) the set $MFm(V)$ of modal formulas $\Phi, \Psi \ldots$, built from atomic modal formulas $N\varphi$, with $\varphi \in Fm(V)$ and $N$ denoting the modality *necessity*, using Lukasiewicz logic and truth-constants $\overline{r}$ for each rational $r \in [0,1]$.

Axioms and rules of $FN(Ł_n^+, RPL)$ are the axioms of $Ł_n^+$ for non-modal formulas, the axioms of RPL for modal formulas, plus the following possibilistic states related axioms

$(FN1)$ $\neg N\bot$,
$(FN2)$ $N(\varphi \to \psi) \to (N\varphi \to N\psi)$,
$(FN3)$ $N(\varphi \wedge \psi) \equiv (N\varphi \wedge N\psi)$,
$(FN4)$ $N(\overline{r} \oplus \psi) \equiv \overline{r} \oplus N\psi$, for each $r \in \{0, 1/n, \ldots, (n-1)/n, 1\}$.

The rules of inference are modus ponens (for modal and non-modal formulas) and the rule of necessitation: from $\varphi$ derive $N\varphi$

The semantics of $FN(L_n^+, RPL)$ is given by weak and strong possibilistic Kripke models. A weak possibilistic Kripke model is a system $\mathcal{M} = (W, e, I)$ where:

- $W$ is a non-empty set whose elements are called *nodes*,
- $e : W \times V \to \{0, 1/n, \dots, (n-1)/n, 1\}$ is such that, for each $w \in W$, $e(w, \cdot)$ is an evaluation of propositional variables which extends to a Ł$_n^+$-evaluation of (non-modal) formulas of $Fm(V)$ in the usual way.
- $I$ is a possibilictic-state on the MV-algebra over $Fm_W = \{u : W \to \{0, 1/n, .., 1\}\}$ with the point-wise extensions of the operations.

Given a weak possibilistic Kripke model $\mathcal{M}$ for $FP(L_n^+, RPL)$, a formula $\Phi$ and a $w \in W$, the truth value of $\Phi$ in $\mathcal{M}$ at the node $w$, denoted $\|\Phi\|_{\mathcal{M},w}$, is inductively defined as follows:

- If $\Phi$ is a non-modal formula $\varphi$, then $\|\varphi\|_{\mathcal{M},w} = e(w, \varphi)$,
- If $\Phi$ is an atomic modal formula $P(\psi)$, then $\|P(\psi)\|_{\mathcal{M},w} = I(u_\varphi)$, where $u_\varphi$ is defined by putting $u_\varphi(w) = e(w, \varphi)$ for all $w \in W$.
- If $\Phi$ is a non-atomic modal formula, then its truth value is computed by evaluating its atomic modal sub-formulas, and then by using the truth functions associated to the $L$-connectives occurring in $\Phi$.

A strong possibilistic Kripke model is a system $\mathcal{N} = (W, e, \pi)$ where $W$ and $e$ are defined as in the case of a weak possibilistic Kripke model and $\pi$ is a possibility distribution on $W$, i.e. $\pi : W \to \{0, 1/n, .., 1\}$ satisfying $\max_{w \in W} \pi(w) = 1$. Evaluations of formulas of $FN(Ł_n^+, RPL)$ in a strong possibilistic Kripke model $\mathcal{N}$ are defined as in the case of weak model except for the case of atomic modal formulas:

- If $\Phi$ is an atomic modal formula $N(\psi)$, then $\|N\psi\|_{\mathcal{N}} = \inf_{w \in W} \pi(w) \Rightarrow e(w, \psi)$.

Then we can show:

**Theorem**. The logic $FN(Ł_n^+, RPL)$ is sound and (finite) strongly complete with respect to the class of both weak and strong possibilistic Kripke models.

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Open University of Catalonia
*E-mail address*: enrico@iiia.csic.es