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LATTICE EFFECT ALGEBRAS POSSESING TWO-VALUED STATES

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Common generalizations of MV-algebras [2] and orthomodular lattices are lattice effect algebras [3]. An *effect algebra* $(E; \oplus, 0, 1)$ is a set E with two special elements $0, 1$ and a partial binary operation $\oplus$ which is commutative and associatiove at which these equalities hold if one of their sides exists. Moreover, to every element $a \in E$ there exists a unique element $a' \in E$ with $a \oplus a' = 1$ and if $a \oplus 1$ exists then $a = 0$. In every effect algebra we can define a partial order by $a \leq b$ iff there exists $c \in E$ with $a \oplus c = b$ (we set $c = b \ominus a$). If $(E; \leq)$ is a lattice (a complete lattice) then $(E; \oplus, 0, 1)$ is called a *lattice effect algebra* (*a complete lattice effect algebra*).

Generalized effect algebras as posets are unbounded versions of effect algebras. In this case instead of the axiom on the existence of a' with $a \oplus a' = 1$ for all $a \in E$ we have cancellation law, i.e., $a \oplus b = a \oplus c$ implies $b = c$ and, moreover, $a \oplus b = 0$ implies $a = b = 0$.

A well known fact is that every generalized effect algebra P can be uniquely extended onto effect algebra E (called an *effect algebraic extension of* P) in which P is an order ideal in E and $P^* = E \setminus P$ is a dual poset to P. We write $E = P \dot{\cup} P^*$ (a disjoint union) [4]. On the other hand not every (lattice) effect algebra E becomes this way. We can prove

Theorem 1. [6] *Let $(E; \oplus, 0, 1)$ be an effect algebra. The following conditions are equivalent:*

(i) *There exists a two-valued state ω on E.*

(ii) *There exists a sub-generalized effect algebra P_ω of E such that $E = P_\omega \dot{\cup} P_\omega^*$, where $P_\omega^* = \{1 \ominus x \mid x \in P_\omega\}$, $P_\omega \cap P_\omega^* = \emptyset$ and P_ω is an order ideal in E.*

In this case $P_\omega = \omega^{-1}(\{0\})$.

For Archimedean atomic lattice effect algebra E we can show a sufficient condition (F) for the existence of a two-valued state ω on E:
(F) There exists a finite set $F = \{p_\kappa \mid \kappa \in H\}$ of pairwise noncompatible atoms of E such that for every atomic block M of E there exists $\kappa_M \in H$ such that $p_{\kappa_M} \in C(M)$ and $\omega(p_{\kappa_M}) = 1$. [6]

Moreover, we can prove that this condition (F) is a necessary and sufficient condition for the existence of a two-valued state ω on: (a) every (o)-continuous Archimedean atomic lattice effect algebra, (b) every block-finite Archimedean atomic lattice effect algebra. Here a *block* of a lattice effect algebra is a maximal sub-lattice effect algebra being an MV-algebra. A lattice effect algebra is called *block-finite* if it has only finitely many blocks.

Using this facts we can prove

Theorem 2. [7] *Every Archimedean atomic lattice effect algebra with at most five blocks possess a state.*

Note that in present time the known example of finite lattice effect algebra admitting no states has nineteen blocks.

Finally, since the existence of a two-valued state ω on an effect algebra E is equivalent to the fact that E is an effect algebraic extension of a sub-generalized effect algebra $P_\omega = \{x \in E \mid \omega(x) = 0\}$, the following question arises:

If F_1 and F_2 are two sets of pairwise noncompatible atoms of an Archimedean atomic lattice effect algebra E and ω_1, ω_2 are two-valued states on E at which $P_{F_1} = \omega_1^{-1}(\{0\})$ and $P_{F_2} = \omega_2^{-1}(\{0\})$ are sub-generalized effect algebras of E with $E = P_{F_1} \dot{\cup} P^*_{F_1} = P_{F_2} \dot{\cup} P^*_{F_2}$ whether (or at which conditions) P_{F_1} and P_{F_2} are isomorphic generalized effect algebras.

We show that two non-isomorphic generalized effect algebras P_{F_1} and P_{F_2} may have a common (or isomorphic) effect algebraic extension.

In spite of the fact that an atomic lattice effect algebra E may have nonatomic block [1] the following statement can be proved:

Theorem 3. [5] *Let E be an Archimedean atomic lattice effect algebra and F_1, F_2 are two sets of pairwise noncompatible atoms satisfying the condition (F). Let there exists a bijection $\psi : F_1 \to F_2$ such that, for every atomic block M of E we have $a \in M \cap F_1 \iff \psi(a) \in M \cap F_2$. Then $P_{F_1} \cong P_{F_2}$.*

This very simple condition is not necessary for the isomorphism of P_{F_1} and P_{F_2}. The necessary and sufficient condition for P_{F_1} and P_{F_2} is the following:

Theorem 4. [5] *Let E_1, E_2 be Archimedean atomic lattice effect algebras, $F_1 \subseteq E_1, F_2 \subseteq E_2$ are two sets of pairwise noncompatible atoms satisfying the condition (F). Let $\mathcal{M}_1, \mathcal{M}_2$ are families of all atomic blocks of E_1 and E_2 respectively. Then the condition* (i) *implies the conditions* (ii) *and* (iii)*:*

(i) *P_{F_1} and P_{F_2} are isomorphic generalized effect algebras.*
(ii) *There exist bijections $\psi : \mathcal{A}t(E_1) \to \mathcal{A}t(E_2)$ and $\alpha : \mathcal{M}_1 \to \mathcal{M}_2$ such that for any $M \in \mathcal{M}_1$ there exists an isomorphism $\varphi_M : M \to \alpha(M)$ of atomic MV-effect algebras such that $\varphi_M(a) = \psi(a)$ for all $a \in \mathcal{A}t(M)$ and $\psi(F_1) = F_2$.*
(iii) *There is a bijection $\psi : \mathcal{A}t(E_1) \to \mathcal{A}t(E_2)$ such that*
   (a) *$\psi(F_1) = F_2$.*
   (b) *$ord(\psi(a)) = ord(a)$ for all $a \in \mathcal{A}t(E_1)$.*
   (c) *$a \longleftrightarrow b$ iff $\psi(a) \longleftrightarrow \psi(b)$, for all atoms $a, b \in \mathcal{A}t(E_1)$.*

Moreover, any of the conditions (i) *and* (ii) *implies the condition* (iii) *and if E_1 and E_2 are complete effect algebras then all the conditions* (i), (ii) *and* (iii) *are mutually equivalent.*

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