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Coauthors: E. Vinceková

IDEALS IN MV PAIRS

S. PULMANNOVA

The concept of an MV-algebra was introduced by Chang [4] as an algebraic basis for many-valued logic. It turned out that MV-algebras are a subclass of a more general class of effect algebras [7, 6]. Namely, MV-algebras are in one-to-one correspondence with lattice ordered effect algebras satisfying the Riesz decomposition property [2], the latter are called MV-effect algebras.

In the study of congruences and quotients of effect algebras, a crucial role is played by so-called Riesz ideals [13, 8]. Namely, every Riesz ideal gives rise to a congruence, and if a congruence is generated by an ideal, then this ideal must be Riesz [8, 5], but there are congruences which are not induced by any ideal [1]. In effect algebras satisfying Riesz decomposition properties (and boolean algebras as well as MV-algebras belong to this class), every ideal is a Riesz ideal. In addition, every (effect algebra) ideal in MV-algebras is an MV-algebra ideal, and the corresponding congruence is an MV-algebra congruence, in particular, the quotient is an MV-algebra. Similar situation is in boolean algebras. On the other hand, not every effect algebra congruence in the latter structures is an MV-algebra (boolean algebra) congruence. It is well-known that every congruence on effect algebras preserves the Riesz decomposition properties, but not necessarily the lattice structure.

An important relation between MV-algebras and boolean algebras is obtained taking into account that every MV-algebra admits a structure of a bounded distributive lattice. Namely, let us now recall the concept of a boolean algebra R-generated by a bounded distributive lattice D. We say that D R-generates a boolean algebra $B(D)$ iff it is its 0,1-sublattice and generates it as a boolean algebra. G. Jenča in his recent work [10] showed that when the lattice D is an MV-effect algebra, then there exists a surjective morphism of effect algebras $\psi_D : B(D) \to D$ and $B(D)/\sim_{\psi_D}$ is isomorphic to D ([10]). In [9], the question is solved, if we can express the morphism ψ_D in terms of boolean algebras only, without using the structure of effect algebra. The answer in [9, Th. 4.1, Th. 3.9] says, that for every MV-effect algebra M, there exists a group $G(M)$ (subgroup of the automorphism group of $B(M)$) such that an equivalence relation on $B(M)$ associated with $G(M)$ equals $\sim_{\psi_M}$ and vice versa, under some special conditions on the group G, a pair (B, G) (BG-pair), produces an MV-effect algebra $B/\sim_G$. The condition, or the special property inflicted on G, is that the BG-pair must be a so called *MV-pair*. Namely, a BG-pair (B, G) is called an MV-pair iff the following conditions are satisfied:

(MVP1) For all $a, b \in B, f \in G$ such that $a \leq b$ and $f(a) \leq b$, there is $h \in G$ such that $h(a) = f(a)$ and $h(b) = b$.

(MVP2) For all $a, b \in B$ and $x \in L(a, b)$, there exists $m \in max(L(a, b))$ with $m \geq x$, where $L(a, b) = \{a \wedge f(b) : f \in G\}$ and $\max(L(a, b))$ is the set of all maximal elements in $L(a, b)$.

Recently, it was proved by Jenča that, given an MV-pair (B, G), the quotient $B/{\sim_G}$, where $\sim_G$ is an equivalence relation naturally associated with G, is an MV-algebra, and conversely, to every MV-algebra there corresponds an MV-pair.

In this paper, we study relations between congruences of B and congruences of $B/{\sim_G}$ induced by a G-invariant ideal I of B. In addition we bring some relations between ideals in MV-algebras and in the corresponding R-generated boolean algebras. Our interest turns to the MV-pair property and the question is, if having an MV-pair (B, G), we get again an MV-pair $(B/I, G')$ for a boolean algebra ideal I. If it was the case, then by the Theorem 3.9. in [9], $(B/I)/\sim_{G'}$ is again an MV-effect algebra and another question may arise – is it the same structure as if we do the process in the reversed order, that is, get an MV-effect algebra $B/\sim_G$ and factorize it by an ideal $I/\sim_G$? We answer these questions mainly affirmatively.We also study relations between ideals in MV-algebras and in the corresponding R-generated boolean algebras. We get, for MV-effect algebras M_1, M_2 and their R-generated boolean algebras, the following commuting diagram, where ϕ is a surjective homomorphism of MV-algebras, ϕ^* is a surjective homomorphism of boolean algebras that extends ϕ and ψ_1, ψ_2 are effect algebra morphisms:

$$\begin{array}{ccc} M_1 & \xleftarrow{\psi_1} & B(M_1) \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi^*} \\ M_2 & \xleftarrow{\psi_2} & B(M_2) \end{array}$$

Finally, we study relations between states on MV-algebras, the corresponding R-generated Boolean algebras and MV-pairs.

References

[1] A. Avallone and P. Vitolo: *Congruences and Ideals of Effect Algebras*, Order **20** (2003), 67-77.

[2] A. Dvurečenskij, S. Pulmannová: *New Trends in Quantum Structures.* Kluwer Academic Publishers, Dordrecht, 2000.

[3] G. Grätzer: *General Lattice Theory.* Birkhäuser, Stuttgart, 1978.

[4] C. Chang: *Algebraic analysis of many-valued logic*, Trans. Amer. Math. Soc. **89**(1959),74-80.

[5] G. Chevalier, S. Pulmannová: *Some Ideal Lattices in Partial Abelian Monoids and Effect Algebras.* Order **17** (2000), 75-92.

[6] F. Chovanec, F. Kôpka: *D-lattices* Inter. J. Theor. Phys. **34** (1995), 1297-1302.

[7] D.J. Foulis and M.K. Bennett:*Effect algebras and unsharp quantum logics*, Found. Phys. **24** (1994), 1325-1346.

[8] S. Gudder and S. Pulmannová: *Quotients of partial abelian monoids*, Algebra univers. **47** (2002), 395-421.

[9] G. Jenča: *A representation theorem for MV-algebras.* Soft Computing, 11(6): 557-564 (2007).

[10] G. Jenča: *Boolena algebras R-generated by MV-effect algebras.* Fuzzy sets and systems **145** (2004), 279–285.

[11] F. Kôpka and F. Chovanec: *D-posets*, Math. Slovaca **44** (1994), 21-34.

[12] S. Koppelberg: *Handbook of Boolean Algebras* North Holland, Amsterdam, 1989.

[13] S. Pulmannová: *Congruences in partial abelian monoids*, Algebra univers.**37** (1997), 119-140.

Mathematical Institute, Slovak Academy of Sciences, Bratoslava, Slovakia
E-mail address: pulmann@mat.savba.sk