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FAMILIES OF ARCHIMEDEAN LATTICE EFFECT ALGEBRAS POSSESSING STATES

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Effect algebras are very natural algebraic structures for to be carriers of states or probability measures when some events are pairwise noncompatible, or unsharp, fuzzy or imprecise. On the other hand there exist even finite effect algebras admitting no states and hence no probability measures [5]. Lattice effect algebras generalize orthomodular lattices and MV-algebras. A lattice effect algebra E is an orthomodular lattice iff every element of E is sharp and E is an MV-algebra (more precisely an MV-effect algebra) iff every pair of elements is compatible. Moreover, in every lattice effect algebra E the set $S(E) = \{x \in E |\ x \wedge x^{'} = 0\}$ of sharp elements is an orthomodular lattice [3]. Further, the notions of a state on orthomodular lattice and on a corresponding lattice effect algebra coincide. Thus the restriction of any state on lattice effect algebra E to the set $S(E)$ of sharp elements is a state on this orthomodular lattice $S(E)$. The natural question arises, whether every state on the set of sharp elements of E can be extended onto whole E. We can show some families of Archimedean atomic lattice effect algebras for which the answer about extension of a state from sharp elements onto whole effect algebra is positive.

A special types of effect algebras called sharply dominating and S-dominating has been introduced by S. Gudder in [1, 2]. Important example is a standard Hilbert spaces effect algebra $E(H)$ of positive linear operators on a complex Hilbert space H which are dominated by identity operator I.

An effect algebra $(E, \oplus, 0, 1)$ is called *sharply dominating* if for every $a \in E$ there exists a smallest sharp element $\hat{a}$ such that $a \leq \hat{a}$. That is $\hat{a} \in S(E)$ and if $b \in S(E)$ satisfies $a \leq b$ then $\hat{a} \leq b$. A *sharply dominating effect algebra* E is called *S-dominating* if $a \wedge p$ exists for every $a \in E$ and $p \in S(E)$.

In next we will use that $S(E)$ is a sub-lattice and a sub-effect algebra of E, [3].

The next theorem gives a characterization of all atomic lattice effect algebras which are sharply dominating and Archimedean. Property (ii) is called "Basic decomposition of elements property". In what follows set $M(E) = \{x \in E |$ if $v \in S(E)$ satisfies $v \leq x$ then $v = 0\}$.

Theorem 1. [6] *Let $(E; \oplus, 0, 1)$ be an atomic lattice effect algebra. The following conditions are equivalent:*

(i) *E is Archimedean and sharply dominating.*
(ii) *For every $x \in E, x \neq 0$ there exists the unique $v_x \in S(E)$, unique set of atoms $\{a_\alpha | \alpha \in \Lambda\}$ and unique positive integers $k_\alpha \neq ord(a_\alpha)$ such that*

$$x = v_x \oplus (\bigoplus\{k_\alpha a_\alpha | \alpha \in \Lambda\}).$$

A net $(a_\alpha)_{\alpha\in\Lambda}$ of elements of a poset $(P, \leq)$ *order converges* to a point $a \in P$ if there are nets $(u_\alpha)_{\alpha\in\Lambda}$ and $(v_\alpha)_{\alpha\in\Lambda}$ of elements of P such that

$$a \uparrow u_\alpha \leq a_\alpha \leq v_\alpha \downarrow a.$$

We write $a_\alpha \xrightarrow{(o)} a$ in P (or briefly $a_\alpha \xrightarrow{(o)} a$). Here $u_\alpha \uparrow a$ means that $u_\alpha \leq u_\beta$ for all $\alpha \leq \beta$ and $a = \bigvee\{u_\alpha | \alpha \in \Lambda\}$. The meaning of $v_\alpha \downarrow a$ is dual.

Recall that a map $\omega : E \to [0, 1]$ is called a (finite additive) *state* on an effect algebra $(E; \oplus, 0, 1)$ if $\omega(1) = 1$ and $x \leq y^{'} \Rightarrow \omega(x \oplus y) = \omega(x) + \omega(y)$. A state is *faithful* if $\omega(x) = 0 \Rightarrow x = 0$. A state ω is called (o)-*continuous (order-continuous)* if $x_\alpha \xrightarrow{(o)} a \Rightarrow \omega(x_\alpha) \to \omega(x)$ for every net $(x_\alpha)_{\alpha\in\Lambda}$ of elements of E.

Theorem 2. [6] *Let $(E; \oplus, 0, 1)$ be a sharply dominating Archimedean atomic lattice effect algebra. Then to every (o)-continuous state ω on $S(E)$ there exists a state $\hat{\omega}$ on E such that $\hat{\omega}|S(E) = \omega$.*

The next example shows that in Theorem 2 the assumption that E is lattice ordered cannot be omitted.

Example 3 The smallest effect algebra E admitting no states has been presented in [5], Example 2.3. Namely, $E = \{0, a, b, c, 2a, 2b, 2c, 3b, 1\}$ and $1 = a \oplus b \oplus c = 3a = 4b = 3c$, which gives $b \oplus c = 2a, a \oplus b = 2c, a \oplus c = 3b$. Thus, E is not lattice ordered, because $a \vee b$ does not exists. Further $S(E) = \{0, 1\}$, hence E is sharply dominating and Archimedean. Nevertheless, there is no state on E extending a state ω existing on $S(E)$.

Finally, let us note that every complete (hence every finite) lattice effect algebra E is Archimedean (see [4], Theorem 3.3). Moreover, E is sharply dominating, because $S(E)$ is a complete sublattice of E and hence $\bigwedge Q \in E$ for all $Q \subseteq S(E)$ (see [3], Theorem 3.7). Further, it is easy to verify that a direct product of Archimedean sharply dominating atomic lattice effect algebras is again an atomic, Archimedean, sharply dominating and lattice ordered effect algebra.

Let $(P; \leq, 0)$ be a poset with zero 0. An element $a^* \in P$ is called a *pseudocomplement of* $a \in P$ if:

(i) $a \wedge_P a^* = 0$,
(ii) $a \wedge_P x = 0 \Rightarrow x \leq a^*$.

If every element of P has a pseudocomplement then P is called a *pseudocomplemented poset with zero.*

Elements of lattices or posetes with zero may have only uniquely determined pseudocomplements. Nevertheless, a pseudocomplementation on a poset P need not be an orthocomplementation on P, namely for quantum structures with unsharp elements (meaning that elements x and $(non)x$, hence x and x' need not be disjoint). We show that for all pseudocomplemented lattice effect algebras E their sets of sharp elements $S(E)$ are sub-lattices of Boolean algebras $P(E) = \{a^* \mid a \in E\}$, on which inherited pseudocomplementation and orthocomplementation coincide. In consequence, for every pseudocomplemented lattice effect algebra E, the set $S(E)$ of its sharp elements is a Boolean algebra which is a sublattice effect algebra of E. Moreover, if E is atomic non-MV-effect algebra then the set $P(E)$ of pseudocomplements is a Boolean algebra with at least one atom and including $S(E)$ as a subalgebra. As an application, we can show some families of lattice effect algebras such that the existence of a pseudocomplementation implies existence of states on them.

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