Measures and topologies on MV-algebras

Hans Weber

My talk is based on article [1]. Our aim is to give a unified topological approach to several results about a certain type of fuzzy measures, namely $T_\infty-$valuations $^{(2)}$ on clans of fuzzy sets $^{(1)}$ (in the sense of Butnariu and Klement [2]). We will transfer the method of [3], used to study FN-topologies $^{(4)}$ and measures on Boolean rings, to the study of $T_\infty-$valuations. So we need, for the domain of the measures, instead of a clan of fuzzy sets a more general structure which is equationally defined and therefore closed with respect to quotients and uniform completions. A structure which satisfies these requirements is that of an MV-algebra.

In this talk I will first mention some algebraic facts about MV-algebras. In particular, it is shown that any σ-complete MV-algebra L is isomorphic to a clan of fuzzy sets, more precisely: Let Ω and $\mathcal{A}$, respectively, be the Stone space and the Stone algebra of the center $C(L)$ of L and $l_\infty(\Omega)$ the space of real-valued bounded functions on Ω; then L can be embedded in the closed linear subspace $\mathcal{L}_\infty(\mathcal{A})$ of $(l_\infty(\Omega), \|\ \|_\infty)$ generated by the characteristic functions χ_A, $A \in \mathcal{A}$. As an immediate consequence one obtains a representation of a measure $^{(3)}$ on L as an integral with respect to a measure on $\mathcal{A}$. Another consequence is the following decomposition theorem: Any complete MV-algebra is isomorphic to a product $L_0 \times \prod_{\alpha \in A} Ł_{n_\alpha}$ where L_0 is an MV-algebra such that $C(L_0)$ is atomless, $n_\alpha \in (\mathbb{N} \cup \{\infty\}) \setminus \{1\}$, $Ł_n$ is a chain of n elements if $n \in \mathbb{N}$ and $Ł_\infty$ is the real interval $[0, 1]$.

Then we study uniform MV-algebras, i.e. MV-algebras L endowed with a uniformity making the operations of L uniformly continuous. It turns out that the uniformity of any uniform MV-algebra can be generated by a family of submeasures. A submeasure η on L is a monotone $[0, \infty]-$valued function on L such that $\eta(0) = 0$ and $\eta(x + y) \leq \eta(x) + \eta(y)$ for all $x, y \in L$; then η induces a semimetric on L by the formula $d(x, y) := \eta(x \vee y - x \wedge y)$ and so (L, d) is a uniform MV-algebra. As a consequence of the algebraic decomposition of complete MV-algebras mentioned above and the known characterization of compact Boolean algebras one obtains e.g. that a uniform MV-algebra is compact if and only if it is (topologically and algebraically) isomorphic to a product $\prod_{\alpha \in A} Ł_{n_\alpha}$, $2 \leq n_\alpha \leq \infty$. A particular role play exhaustive uniform MV-algebras, i.e. uniform MV-algebras such that any monotone sequence is Cauchy. We establish a relationship between these uniformities on an MV-algebra L and the order continuous FN-topologies [4] on the center $C(\widetilde{L})$ of a suitable completion $\widetilde{L}$ of L. This is also used as an important tool to study measures on MV-algebras.

In the last section we study measures on an MV-algebra L with values in a complete Hausdorff locally convex linear space E. We first establish an isomorphism between the space of all $E-$valued exhaustive [5] measures on L and the space of all $E-$valued order continuous measures on a suitable complete Boolean algebra. This isomorphism allows us to transfer results known for measures on Boolean algebras to the case of measures on MV-algebras. In this way we obtain decomposition theorems, Lyapunov's convexity theorem, the Vitali-Hahn-Saks theorem and Nikodým's boundedness theorem for measures on MV-algebras.

(1) A clan $\mathcal{C}$ of fuzzy sets is a lattice of $[0,1]$-valued functions defined on a set Ω containing the constant function 1 such that $f - g \in \mathcal{C}$ whenever $f, g \in \mathcal{C}$ and $g \leq f$.

(2) A $T_\infty-$valuation on $\mathcal{C}$ is a function m on $\mathcal{C}$ satisfying $m(f+g) = m(f)+m(g)$ if $f, g \in \mathcal{C}$ and $f + g \leq 1$.

(3) A measure on an MV-algebra L is a function m on L satisfying $m(f+g) = m(f)+m(g)$ if $f, g \in L$ and $f \leq g'$.

(4) An FN-topology on a Boolean ring R is a ring topology on R such that the multiplication is uniformly continuous.

(5) A measure $m : L \to E$ is exhaustive if $m(x_n)$ converges for any monotone sequence x_n in L.

References

[1] G. Barbieri and H. Weber, *Measures on clans and on MV-algebras*, Handbook of Measure Theory, ed by E. Pap, Elsevier, Amsterdam, 911–945 (2002)

[2] D. Butnariu and E.P. Klement, *Triangular norm based measures and games with fuzzy coalitions*, Kluver, Dordrecht (1993)

[3] H. Weber, *Group- and vector-valued s-bounded contents*, Measure Theory (Oberwolfach 1983) LNM 1089, 181-198 (1984)