Notes on the foundations of fuzzy logic

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Abstract

We discuss two approaches related to the foundational problem of fuzzy logic. The first one starts from a specific fuzzy logic and endows it with a non-standard semantics, with the intention that this semantics can be more easily understood than the canonical one. The second one starts from the application with which fuzzy logic traditionally claims to deal, namely reasoning about properties which are not clearly delimitable.

There has been a lot of discussion around the question on which principles t-norm based many-valued logics, or fuzzy logics for short, are based. A good amount of work deals with the interpretation of truth values (e.g., [HeCa]); further contributions try to justify particular choices of t-norms (e.g., [Law]); and a few approaches address the full framework of a fuzzy logical calculus (e.g., [Fer], [Par2]).

One line of research considers specific fuzzy logics and tries to endow them with alternative semantics (e.g., [Par1]). This amounts to the question if it is possible to make sense out of certain (classes of) residuated lattices. Ideal lattices [WaDi] gave originally rise to the notion, but seem to be unrelated to the present context. A more interesting approach can be found in [OnKo], where certain residuated lattices are represented in the power set of a suitable po-monoid, the monoidal operation being pointwise defined. This approach is particularly appealing if the underlying po-monoid is of a simple form, like for instance a Boolean algebra. Following these lines a semantics for Łukasiewicz logic can in fact be defined [Vet1]. A generalisation is even possible for Basic Logic, although with some loss of elegancy [Vet2].

Interpretations of this kind might open new fields of applications for fuzzy logics – logics whose mathematical theory is incredibly rich and fascinating. Still, the problem how fuzzy logics and their actually intended applications are precisely correlated remains a challenge. It seems to be well worth to change the perspective and to start from the side of intended applications.

I would like to mention a formalism which is apparently not in the scope of fuzzy logics: the logic of approximate reasoning [Rus, DPEGG]. The logic LAE [EGRV] interprets propositions as subsets of a metric space, and it allows to conclude from one proposition to another one even if the former is only approximately a subset of the latter. Namely, the statement $\alpha \overset{t}{\Rightarrow} \beta$ holds under some evaluation v if $v(\alpha) \subseteq U_t(v(\beta))$, where U_t assigns the t-neighborhood to some subset of the space.

According to a common understanding of fuzzy sets, a truth value measures the similarity to the nearest prototype (see, e.g., [DOP]). Indeed, a fuzzy set can, in the typical case, identified with its kernel and a pseudometric; the latter assigns to each point of the base set its distance to the kernel. This idea leads to a setting which is actually not far from the framework of LAE. We shall explore the relationship to a certain extent.

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